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The Chebyshev's property of certain hyperelliptic integrals of the first kind

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ABSTRACT

In this work, we study the Chebyshev's property of the 3-dimensional vector space $E =< J_0, J_1, J_2 >$, where $J_i(h) = \int_{H=h} x^i \frac{dx}{y}$ and $H(x, y) = \frac{1}{2}y^2 + V(x)$ is a hyperelliptic Hamiltonian of degree 7. Our main result asserts that in two specific cases, namely (a) $V'(x) = x^3(1-x)^3$ and (b) $V'(x) = x^5(x-1)$, E is an extended complete Chebyshev space. To this end we use the criterion and the tools developed by Grau et al. in [6]. We pose also the conjecture that E is also a Chebyshev space when $V'(x) = x(x-1)^5$. In this regard we give a partial result, Theorem 1.4, concerning the Chebyshev property of two subspaces of E. To prove it we use another criterion by Mañosas and Villadelprat [7] to study when a collection of Abelian integrals is Chebyshev with accuracy k.

1. Introduction and statement of the results

The paper deals with a problem, closely related to the second part of the 16th Hilbert problem, also known as the “weakened 16th Hilbert problem” or “Hilbert–Arnold problem”: find an upper bound to the number of the zeros of complete Abelian integrals, as functions of parameters. A specific case is to study zeros of hyperelliptic integrals of first kind, stated explicitly by Arnold (problem 7 in [1]). This problem is as follows:

The Arnold 7th problem: [1] Consider a deformation of a planar Hamiltonian system

$$dH + \varepsilon(f dx + g dy) = 0, \tag{1}$$

where f and g are polynomials in x, y of degrees at most n, $\varepsilon > 0$ is a small parameter. The Abelian integrals associated to the deformation (1) are defined as

$$I(h) = \oint_{L_h} f(x, y)dx + g(x, y)dy,$$

where L_h is the oval of the level curve $H(x, y) = h$, and hence

$$L_h \subset \{(x, y) \in \mathbb{R}^2 \ : \ H(x, y) = h, \ h_1 < h < h_2\}.$$

The main question is that how many isolated zeros the Abelian integrals $I(h)$ can have on the considered interval if f, g and H are polynomials with known degrees? And that for the complete hyperelliptic integrals of the first kind

$$J(h) = \oint_{L_h} \frac{\alpha_0 + \alpha_1 x + \cdots + \alpha_{g-1}x^{g-1}}{y} dx,$$

does the g-dimensional family of $J(h)$ forms a Chebyshev family on a real domain? Here, $H(x, y) = y^2 + V(x)$, where $\deg V(x) = 2g + 1 > 4$, and α_i are real parameters for $i = 0, 1, \ldots, g-1$.

Remark 1.1. Recall that a real vector space E of real analytic functions defined on some real interval $\mathbb{I}$ is said to be Chebyshev, provided that each f in E has at most $dimE - 1$

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isolated zeros (counted with multiplicity) on $\mathbb{I}$, and Chebyshev with accuracy m, if each f in E has at most $dimE + m - 1$ isolated zeros there.

In the elliptic case (the genus $g = 1$) there is only one such integral (up to multiplication by a constant), which does not vanish by the Abel theorem. In the case $g = 2$ we have a 2-dimensional space of such integrals, and partial results are known since 2004 [4], by making use of algebraic geometry. It is important to say, that the results of [4] hold true for hyperelliptic Hamiltonians in a generic position. Similar results, were obtained by Wang et al. [5] in 2012. The authors of [5] use the method of [6] (2011) which is based on elementary analysis and classical result on non-oscillation of functions in a real domain. In the present paper, we apply once again the method of [6] but this time to the case $g = 3$. Both in [5] and the present paper, the results are obtained for special hyperelliptic Hamiltonians, namely with only two critical values. Therefore, the corresponding hyperelliptic curves are not generic.

This work deals with the number of isolated zeros of the Abelian integral

$$J(h) = \oint_{L_h} \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2}{y} dx,$$

where α_i, $i = 0, 1, 2$, are real parameters and L_h is the oval of the level curce $H(x, y) = h$ with $H(x, y)$ the Hamiltonian of the system

$$\dot{x} = y, \quad \dot{y} = -x(x-\alpha)^2(x-\beta)^2(x-1). \tag{2}$$

We consider the two particular cases $(\alpha, \beta) = (0, 1)$ and $(\alpha, \beta) = (0, 0)$. In both cases we prove that the Abelian integrals

$$J_0(h) = \oint_{L_h} \frac{1}{y} dx, \quad J_1(h) = \oint_{L_h} \frac{x}{y} dx, \quad J_2(h) = \oint_{L_h} \frac{x^2}{y} dx,$$

form a Chebyshev system for all the values h in which L_h is a periodic orbit of the Hamiltonian system. This implies that $J(h)$ has at most two isolated zeros (counted with multiplicity) for any value of $(\alpha_0, \alpha_1, \alpha_2)$. The main tool used to prove the Chebyshev property comes from the work [6]. Using our notations and in order to apply the result of [6], we need to show that three algebraic functions $W_{10}(x, z_1)$, $W_{11}(x, z_1)$ and $W_{12}(x, z_1)$ (see page 8 for notation) do not vanish on a real interval $x \in (0, x_r)$. We can prove using Sturm's theorem that $W_{10}(x, z_1)$ and $W_{11}(x, z_1)$ do not vanish on the considered interval. However, in order to prove that the last algebraic function $W_{12}(x, z_1)$ does not vanish, we use some techniques in polynomial algebra. Thus the proof is strict, and hence an analytical proof of the fact that the functions $\{J_0(h), J_1(h), J_2(h)\}$ form a Chebyshev system is given.

Our main results are the following.

Theorem 1.2.

(a) *Given system (2) with $\alpha = 0$ and $\beta = 1$. Then the 3-dimensional real vector space $E =< J_0(h), J_1(h), J_2(h) >$ has the Chebyshev's property in the open interval $\left(-\frac{1}{140}, 0\right)$. Hence, the exact upper bound on the number of isolated zeros of $J(h) = \alpha_0 J_0(h) + \alpha_1 J_1(h) + \alpha_2 J_2(h)$ counted with multiplicities is two in $\left(-\frac{1}{140}, 0\right)$ for all real numbers α_0, α_1 and α_2.*

(b) *Given system (2) with $\alpha = \beta = 0$. Then the 3-dimensional real vector space $E =< J_0(h), J_1(h), J_2(h) >$ has the Chebyshev's property in the open interval $\left(-\frac{1}{42}, 0\right)$. Hence, the sharp upper bound on the number of isolated zeros of the Abelian integral $J(h) = \alpha_0 J_0(h) + \alpha_1 J_1(h) + \alpha_2 J_2(h)$ counted with multiplicities is two for $h \in \left(-\frac{1}{42}, 0\right)$ and this for all real numbers α_0, α_1* and *α_2.*

Fig. 1 represents all possible phase portraits of system (2) for different values of the parameters α and β. In the figure, there are seven cases of topologically different phase portraits for system (2) (Fig. 2).

We make the following conjecture on the case (c).

Conjecture 1.3. When $\alpha = \beta = 1$ in system (2), then the 3-dimensional real vector space $E =< J_0(h), J_1(h), J_2(h) >$ has the Chebyshev's property on the open interval $\left(-\frac{1}{42}, 0\right)$. Hence, the exact upper bound on the number of isolated zeros (counted with multiplicities) of the Abelian integral $J(h) = \alpha_0 J_0(h) + \alpha_1 J_1(h) + \alpha_2 J_2(h)$ is two for $h \in \left(-\frac{1}{42}, 0\right)$ and this for all real numbers α_0, α_1 and α_2.

The second aim of this paper is to show:

Theorem 1.4. *Suppose in system (2) that $\alpha = \beta = 1$. Then*

(i) The 2-dimensional real vector spaces $E_1 =< J_0(h), J_1(h) >$ and $E_2 =< J_0(h), J_2(h) >$ have the Chebyshev's property with accuracy one;

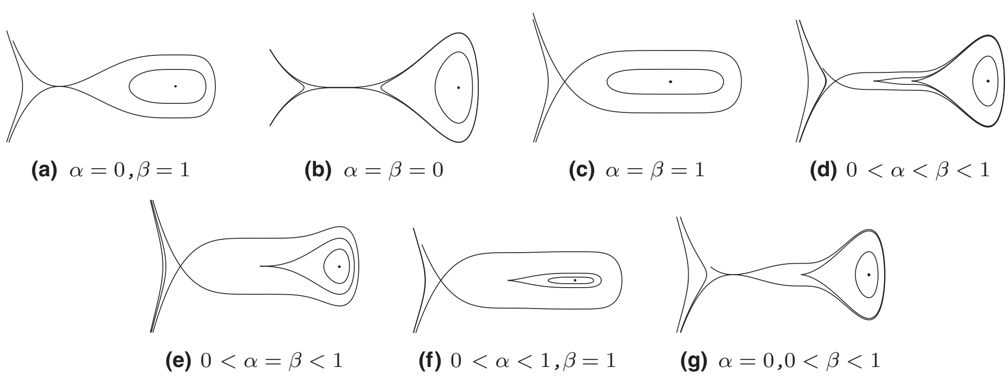

Fig. 1. Phase portraits of system (2) for different values of α and β.

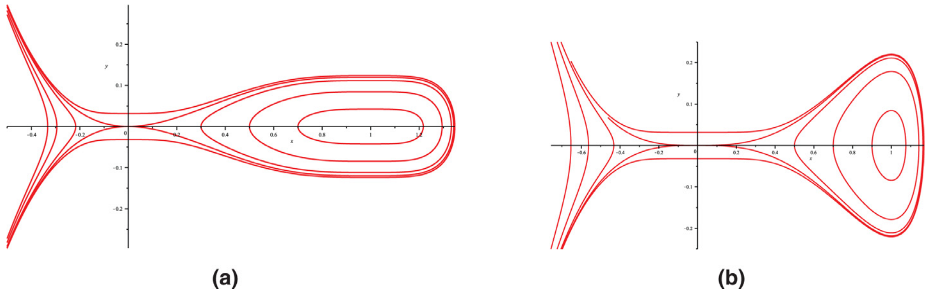

Fig. 2. Phase portraits of system (2) in two cases (a) and (b).

(ii) *The sharp upper bound on the number of isolated zeros of the Abelian integral $J(h) = \alpha_0 J_0(h) + \alpha_1 J_1(h)$, counted with multiplicities, is one for $h \in \left(-\frac{1}{42}, 0\right)$ and this for all real numbers α_0, α_1. Similarly, the exact upper bound on the number of isolated zeros of the Abelian integral $J(h) = \alpha_0 J_0(h) + \alpha_2 J_2(h)$, counted with multiplicities, is one for $h \in \left(-\frac{1}{42}, 0\right)$ and this for all real numbers α_0 and α_2.*

2. Preliminaries

In this section, we will provide some definitions and some lemmas that will be used in the proof of main results.

Definition 2.1. Let $f_0, f_1, \ldots, f_{n-1}$ be real analytic functions on some open interval $\mathbb{I}$ of $\mathbb{R}$. Then

(i) The set $\{f_0, f_1, \ldots, f_{n-1}\}$ is called a Chebyshev space on $\mathbb{I}$ if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on $\mathbb{I}$.

(ii) The set $\{f_0, f_1, \ldots, f_{k-1}\}$ is called a complete Chebyshev space on $\mathbb{I}$ if for all $k = 1, 2, \ldots, n$, any nontrivial linear combination of $(f_0, f_1, \ldots, f_{k-1})$ as

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on $\mathbb{I}$.

(iii) The set $\{f_0, f_1, \ldots, f_{k-1}\}$ is called an extended complete Chebyshev space on $\mathbb{I}$ if for all $k = 1, 2, \ldots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on $\mathbb{I}$ counting multiplicity.

Definition 2.2. Let $f_0, f_1, \ldots, f_{k-1}$ be real analytic functions on some open interval $\mathbb{I}$ of $\mathbb{R}$. The Wronskian of $(f_0, f_1, \ldots, f_{k-1})$ at $x \in \mathbb{I}$ is

$$W[f_0, f_1, \ldots, f_{k-1}](x) = \begin{vmatrix} f_0 & f_1 & \cdots & f_{k-1} \\ f_0' & f_1' & \cdots & f_{k-1}' \\ \vdots & \vdots & \vdots & \vdots \\ f_0^{(k-1)} & f_1^{(k-1)} & \cdots & f_{k-1}^{(k-1)} \end{vmatrix}.$$

Using above definitions, it is clear that if the set $\{f_0, f_1, \ldots, f_{n-1}\}$ is an extended complete Chebyshev space on $\mathbb{I}$, and hence a complete Chebyshev space on $\mathbb{I}$, then it is a Chebyshev space on $\mathbb{I}$. However, the reverse implication is not true.

The following result about an extended complete Chebyshev space is well known (see [9,10] for instance).

Lemma 2.3. *The set $\{f_0, f_1, \ldots, f_{n-1}\}$ is an extended complete Chebyshev space on $\mathbb{I}$ if and only if, for each $k = 1, 2, \ldots, n$,*

$$W[f_0, f_1, \ldots, f_{k-1}](x) \neq 0 \quad \text{for all } x \in \mathbb{I}.$$

We note that the Abelian integral $J_i(h)$ takes the form

$$J_i(h) = \oint_{L_h} x^i y^{2s-1} dx, \quad i = 0, 1, 2,$$

with $s = 0$ and the Hamiltonian function $H(x, y) = A(x) + \frac{1}{2}y^2$ is an analytic function in some open subset of the plane. We assume that it has a local minimum at the origin. The projection of the period annulus on the x-axis is then (x_l, x_r) and also $xA'(x) > 0$ for all $x \in (x_l, x_r)\backslash\{0\}$. Therefore, there exists an involution $z(x)$ with $x_l < z(x) < 0$ such that $A(x) = A(z(x))$ for all $0 < x < x_r$. We recall that a C^1-mapping $z : \mathbb{I} \longrightarrow \mathbb{I}$ is an involution on $\mathbb{I}$ if $z^2 = id.$ and $z \neq id.$ Below we recall Theorem B of [6] and Theorem A of [7], which are essential in our proofs. We point out that the next lemma, Lemma 2.4, is only a particular case of Theorem B in [6].

Lemma 2.4. *Suppose that*

$$J_i(h) = \int_{L_h} f_i(x) y^{2s-1} dx, \qquad i = 0, 1, \ldots, n-1,$$

where, for each $h \in (0, h_0)$, L_h is the oval surrounding the origin inside the level curve $\{A(x) + \frac{1}{2}y^2 = h\}$. Let $z = z(x)$ be the involution associated to $A(x)$, and define

$$\ell_i(x) = \left(\frac{f_i}{A'}\right)(x) - \left(\frac{f_i}{A'}\right)(z(x)), \qquad i = 0, 1, \ldots, n-1.$$

(i) *If $s > n-2$ and $\{\ell_0, \ell_1, \ldots, \ell_{n-1}\}$ is a complete Chebyshev space on $(0, x_r)$, then $\{J_0, J_1, \ldots, J_{n-1}\}$ is an extended complete Chebyshev space on $(0, h_0)$.*

(ii) *If the following conditions are satisfied:*
(1) $W[l_0, \ldots, l_i]$ is non-vanishing on $(0, x_r)$ for $i = 0, 1, \ldots, n-2$.
(2) $W[l_0, \ldots, l_{n-1}]$ has k zeros on $(0, x_r)$ counted with multiplicity, and
(3) $s > n+k-2$,
then the set $\{J_0, J_1, \ldots, J_{n-1}\}$ is a Chebyshev system with accuracy k on the open interval $(0, h_0)$.

The convergence conditions $s > n-2$ and $s > n+k-2$ are not always satisfied, and hence we cannot always apply

Lemma 2.4 directly. To overcome this problem we can use the next result to increase the power of y by 2 in each step.

Lemma 2.5 ([6]). *Let L_h be an oval inside the level curve $A(x) + \frac{1}{2}y^2 = h$ and consider a function F such that $\frac{F}{A'}$ is analytic at $x = 0$. Then, for each $k \in \mathbb{N}$,*

$$\oint_{L_h} F(x)y^{k-2}dx = \oint_{L_h} G(x)y^k dx,$$

where $G(x) = \frac{1}{k}(\frac{F}{A'})'(x)$.

3. Proof of Theorem 1.2

In this section, based on the above arguments, we will prove the results in Theorem 1.2.

Case (a). When $\alpha = 0,\ \beta = 1$, then system (2) becomes

$$\dot{x} = y, \quad \dot{y} = -x^3(x-1)^3, \tag{3}$$

with the Hamiltonian function

$$H_1(x,y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{3}{5}x^5 - \frac{1}{2}x^6 + \frac{1}{7}x^7,$$

and with the continuous family of ovals $L_1(h)$ surrounding the nilpotent center (1,0), where

$$L_1(h) \subset \left\{(x,y) \in \mathbb{R}^2 : H_1(x,y) = h, -\frac{1}{140} < h < 0\right\}.$$

Therefore, we will study the following complete hyperelliptic integral of the first kind

$$J_1(h) = \oint_{L_1(h)} \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2}{y} dx, \quad -\frac{1}{140} < h < 0, \quad \alpha_i \in \mathbb{R}, \quad i = 0,1,2.$$

As the origin is not the local minimum of $H_1(x,y)$, we shift the nilpotent center (1,0) of system (3) to the origin by the transformation $x = 1 - u, y = -v$ and still denote the pair variable by (x, y) after applying the transformation for simplicity. Therefore, system (3) turns into

$$\dot{x} = y, \quad \dot{y} = x^3(x-1)^3,$$

which has the Hamiltonian function $H_1^*(x,y) = A_1(x) + \frac{1}{2}y^2$ with a local minimum at the origin and the continuous family of ovals $L_1^*(h)$ surrounding the center (0,0) where

$$A_1(x) = \frac{1}{4}x^4 - \frac{3}{5}x^5 + \frac{1}{2}x^6 - \frac{1}{7}x^7.$$

Let us consider the Abelian integral $J_1(h)$ with the Hamiltonian function $H_1^*(x,y)$, which is a linear combination of $\{J_{10}(h),\ J_{11}(h),\ J_{12}(h)\}$, where $J_{1i}(h) = \oint_{L_1^*(h)} x^i y^{2s-1} dx$, $i = 0,1,2$ with $s = 0$. And that

$$L_1^*(h) \subset \left\{(x,y) \in \mathbb{R}^2 : H_1^*(x,y) = h,\ 0 < h < \frac{1}{140}\right\}$$

is the period annulus surrounding the nilpotent center bounded by the homoclinic loop $H_1^*(x,y) = \frac{1}{140}$. We see that the projection of the period annulus on the x-axis is $(x_l, 1)$, where

$$x_l = -\frac{1}{30}\frac{(350 + 105\sqrt{15})^{\frac{2}{3}} - 35 + 5(350 + 105\sqrt{15})^{\frac{1}{3}}}{(350 + 105\sqrt{15})^{\frac{1}{3}}} \in \left(-\frac{351}{1024}, -\frac{175}{512}\right),$$

and that $xA_1'(x) > 0$ for all $x \in (x_l, 1)\backslash\{0\}$. Therefore, there exists an invertible function $z_1(x)$ with $x_l < z_1(x) < 0$ such that $A_1(x) = A_1(z_1(x))$ for $0 < x < 1$ (see Fig. 3). But in this case $n = 3$ and $s = 0$, so one of the hypotheses (or assumptions) of Lemma 2.4, i.e. $s > n - 2$ is violated. However, it is possible to overcome this problem, using Lemma 2.5, and obtain new Abelian integrals for which the corresponding s is large enough to verify the inequality. Here we have to increase the power s to three so that the condition $s > n - 2$ holds. In order to apply Lemma 2.4 we follow the approach in [6], that consists in multiplying by $A(x) + \frac{1}{2}y^2$ the 1-form and dividing the Abelian integral by h.

Hence, on the oval $L_1^*(h)$, we can write

$$\begin{aligned} J_{1i}(h) &= \frac{1}{h}\oint_{L_1^*(h)} \Big(A_1(x) + \frac{1}{2}y^2\Big)x^i y^{-1} dx \\ &= \frac{1}{2h}\left(\oint_{L_1^*(h)} 2x^i A_1(x) y^{-1} dx + \oint_{L_1^*(h)} x^i y dx\right), \\ i &= 0,1,2. \end{aligned} \tag{4}$$

Now, we apply Lemma 2.5 with $k = 1$ and $F(x) = 2x^i A_1(x)$ to the first integral above to get

$$\oint_{L_1^*(h)} 2x^i A_1(x) y^{-1} dx = \oint_{L_1^*(h)} G_{1i}(x) y dx,$$

where $G_{1i}(x) = \frac{d}{dx}\left(\frac{2x^i A_1(x)}{A_1'(x)}\right) = \frac{g_{1i}(x)}{70(x-1)^4}$ with

$$\begin{aligned} g_{1i}(x) = x^{3+i}&(20(i+1)x - (90i + 80) + (154i + 126)x^{-1} \\ &-(119i + 98)x^{-2} + 35(i+1)x^{-3}). \end{aligned}$$

By (4) we obtain

$$\begin{aligned} J_{1i}(h) &= \frac{1}{2h}\oint_{L_1^*(h)} \big(x^i + G_{1i}(x)\big) y dx \\ &= \frac{1}{4h^2}\oint_{L_1^*(h)} (2A_1(x) + y^2)(x^i + G_{1i}(x)) y dx \\ &= \frac{1}{4h^2}\left(\oint_{L_1^*(h)} 2A_1(x)(x^i + G_{1i}(x)) y dx \right. \\ &\quad \left. + \oint_{L_1^*(h)} (x^i + G_{1i}(x)) y^3 dx\right). \end{aligned} \tag{5}$$

Again we apply Lemma 2.5 with $k = 3$ and $F(x) = 2(x^i + G_{1i}(x))A_1(x)$ to the first integral above to get

$$\oint_{L_1^*(h)} 2(x^i + G_{1i}(x))A_1(x) y dx = \oint_{L_1^*(h)} H_{1i}(x) y^3 dx,$$

where $H_{1i}(x) = \frac{1}{3}\frac{d}{dx}\left(\frac{2(x^i + G_{1i}(x))A_1(x)}{A_1'(x)}\right) = \frac{h_{1i}(x)}{14700(x-1)^8}$ with

$$\begin{aligned} h_{1i}(x) = &(3675 + 4900i + 1225i^2)x^i \\ &+ (-22050 - 32585i - 8330i^2)x^{i+1} \\ &+ (64386 + 98245i + 24941i^2)x^{i+2} \\ &- (115248 + 174356i + 42952i^2)x^{i+3} \\ &+ (134778 + 197946i + 46536i^2)x^{i+4} \\ &- (103740 + 146370i + 32480i^2)x^{i+5} \\ &+ (50820 + 68520i + 14260i^2)x^{i+6} \\ &- (14400 + 18500i + 3600i^2)x^{i+7} \\ &+ (1800 + 2200i + 400i^2)x^{i+8}. \end{aligned}$$

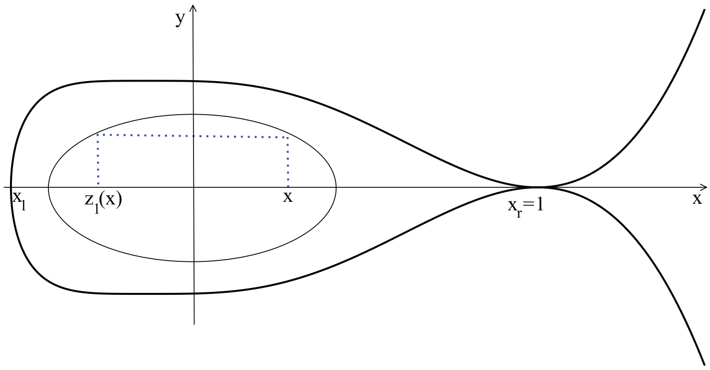

Fig. 3. The involution $z_1(x)$ defined by $A_1(x) = A_1(z_1(x))$.

From (5) we obtain

$$4h^2 J_{1i}(h) = \oint_{L_1^*(h)} f_{1i}(x) y^3 dx \equiv \tilde{J}_{1i}(h),$$

where $f_{1i}(x) = x^i + G_{1i}(x) + H_{1i}(x)$. It is clear that $\{\tilde{J}_{10}(h), \tilde{J}_{11}(h), \tilde{J}_{12}(h)\}$ is an extended complete Chebyshev space on $\left(0, \frac{1}{140}\right)$ if and only if $\{J_{10}(h), J_{11}(h), J_{12}(h)\}$ is as well. As the condition $s > n - 2$ is now satisfied, then we can use Lemma 2.4. Thus, by setting

$$\ell_{1i}(x) = \left(\frac{f_{1i}}{A_1'}\right)(x) - \left(\frac{f_{1i}}{A_1'}\right)(z_1(x)),$$

we need to check that $\{\ell_{10}(x), \ell_{11}(x), \ell_{12}(x)\}$ is a complete Chebyshev space on $x \in (0, 1)$. We only need to prove that the Wronskians $W[\ell_{10}](x)$ of ℓ_{10}, $W[\ell_{10}, \ell_{11}](x)$ of ℓ_{10}, ℓ_{11} and $W[\ell_{10}, \ell_{11}, \ell_{12}](x)$ of $\{\ell_{10}, \ell_{11}, \ell_{12}\}$ are non-vanishing on the interval (0,1).

The involution $z_1 = z_1(x)$ for $x \in (0, 1)$ verifies that

$$A_1(x) - A_1(z_1(x)) = -\frac{1}{140}(x - z_1) q_1(x, z_1) = 0,$$

where

$$\begin{aligned} q_1(x, z_1) = {} & 20x^6 - 70x^5 + 20z_1x^5 + 84x^4 - 70z_1x^4 \\ & + 20{z_1}^2x^4 - 35x^3 + 84z_1x^3 - 70{z_1}^2x^3 \\ & + 20x^3{z_1}^3 - 35z_1x^2 + 84{z_1}^2x^2 - 70x^2{z_1}^3 \\ & + 20x^2{z_1}^4 - 35{z_1}^2x + 84x{z_1}^3 - 70x{z_1}^4 \\ & + 20x{z_1}^5 - 35{z_1}^3 + 84{z_1}^4 - 70{z_1}^5 + 20{z_1}^6. \end{aligned}$$

Using Maple, we find that

$$\begin{aligned} W[\ell_{10}](x) &= \frac{(x - z_1) W_{10}(x, z_1)}{4900(x-1)^{11} x^3 (z_1 - 1)^{11} z_1^3}, \\ W[\ell_{10}, \ell_{11}](x) &= \frac{(x - z_1)^3 W_{11}(x, z_1)}{z_1^6 (z_1 - 1)^{22} x^6 (x-1)^{22} W_{01}(x, z_1)}, \\ W[\ell_{10}, \ell_{11}, \ell_{12}](x) &= \frac{(x - z_1)^6 W_{12}(x, z_1)}{z_1^9 (z_1 - 1)^{33} x^9 (x-1)^{33} W_{01}^3(x, z_1)}, \end{aligned}$$

where $W_{1i}(x, z_1)$, $i = 0, 1, 2$, are polynomials in (x, z_1) with long expressions, and

$$\begin{aligned} W_{01}(x, z_1) = {} & 60z_1^2x^3 + 168z_1x^2 - 210z_1^2x^2 \\ & + 80x^2z_1^3 - 70z_1x + 252z_1^2x - 280xz_1^3 \\ & + 100xz_1^4 + 40z_1x^4 - 140z_1x^3 - 105z_1^2 \\ & + 336z_1^3 - 350z_1^4 + 120{z_1}^5 \\ & + 20x^5 - 70x^4 + 84x^3 - 35x^2. \end{aligned}$$

The resultant with respect to z_1 between $W_{01}(x, z_1)$ and $q_1(x, z_1)$ is

$$\begin{aligned} p_{01}(x) = {} & 53782400000x^6(20x^3 - 70x^2 + 84x - 35)^3 \\ & \times (20x^3 + 10x^2 + 4x + 1)^3 (x - 1)^6. \end{aligned}$$

It is easy to see that $p_{01}(x)$ does not vanish on (0, 1). This implies that $W[\ell_{10}, \ell_{11}](x)$ and $W[\ell_{10}, \ell_{11}, \ell_{12}](x)$ are well defined in the domain $x_l < z_1(x) < 0 < x < 1$.

In order to determine whether these three Wronskians can have zeros on (0, 1), we shall rely on the symbolic computations by Maple to compute the resultant of $W_{1i}(x, z_1)$, $i = 0, 1, 2$ and $q_1(x, z_1)$ with respect to z_1, and then apply Sturm's Theorem to verify nonexistence of zeros of $p_{1i}(x)$, $i = 0, 1, 2$, in (0, 1) where $p_{1i}(x)$, $i = 0, 1, 2$ are polynomials of high degree in x which will be introduced below.

The resultant of $q_1(x, z_1)$ and $W_{10}(x, z_1)$ with respect to z_1 is

$$R_{10}(q_1, W_{10}, z_1) = (x - 1)^{30} p_{10}(x),$$

where $p_{10}(x)$ is a polynomial of degree 96 in x. Using Sturm's Theorem, we get $p_{10}(x) \neq 0$ for all $x \in (0, 1)$. Thus, $W_{10}(x, z_1) = 0$ and $q_1(x, z_1) = 0$ have no common roots. This fact implies that $W[l_{10}](x) \neq 0$ for all $x \in (0, 1)$.

The resultant with respect to z_1 between $q_1(x, z_1)$ and $W_{11}(x, z_1)$ is

$$R_{11}(q_1, W_{11}, z_1) = (x - 1)^{64} p_{11}(x),$$

where $p_{11}(x)$ is a polynomial of degree 212 in x. By applying Sturm's Theorem again, we obtain that $p_{11}(x) \neq 0$ for all $x \in (0, 1)$. Thus, $W_{11}(x, z_1) = 0$ and $q_1(x, z_1) = 0$ have no common roots, and this implies that $W[l_{10}, l_{11}](x) \neq 0$ for all $x \in (0, 1)$.

The resultant with respect to z_1 between $q_1(x, z_1)$ and $W_{12}(x, z_1)$ is

$$R_{12}(q_1, W_{12}, z_1) = (x - 1)^{102} p_{121}(x),$$

where $p_{121}(x)$ is a polynomial of degree 348 in x. By applying Sturm's Theorem, we get however that $p_{121}(x)$ has a unique zero in the open interval (0, 1). Moreover, by using function realroot with accuracy $\frac{1}{10000}$ in Maple, we see that this root lies in the closed subinterval $[\frac{10797}{16384}, \frac{5399}{8192}]$ of (0, 1).

For further study, we calculate the resultant of $q_1(x, z_1)$ and $W_{12}(x, z_1)$ with respect to x. Doing so we obtain

$$R_{12}(q_1, W_{12}, x) = (z_1 - 1)^{102} p_{122}(z_1),$$

where $p_{122}(z_1)$ is a polynomial of degree 348 in z_1. By applying Sturm's Theorem, we find that $p_{122}(z_1)$ has three real roots in $(x_l, 0)$. By using function realroot with accuracy $\frac{1}{10000}$ in Maple, the three roots are in the closed subintervals $[-\frac{199}{16384}, -\frac{99}{8192}]$, $[-\frac{4079}{16384}, -\frac{2039}{8192}]$ and $[-\frac{1391}{4096}, -\frac{5563}{16384}]$ contained in $(x_l, 0)$.

Therefore, if the equations $q_1(x, z_1) = 0$ and $W_{12}(x, z_1) = 0$ have one common root $(x^*, z_1^*) \in (0, 1) \times (x_l, 0)$, the point (x^*, z_1^*) must be in the following three domains

$$D_1 := \left\{(x, z_1) | \frac{10797}{16384} \leqslant x \leqslant \frac{5399}{8192}, \ -\frac{199}{16384} \leqslant z_1 \leqslant -\frac{99}{8192}\right\},$$
$$D_2 := \left\{(x, z_1) \mid \frac{10797}{16384} \leqslant x \leqslant \frac{5399}{8192}, \ -\frac{4079}{16384} \leqslant z_1 \leqslant -\frac{2039}{8192}\right\},$$
$$D_3 := \left\{(x, z_1) \mid \frac{10797}{16384} \leqslant x \leqslant \frac{5399}{8192}, \ -\frac{1391}{4096} \leqslant z_1 \leqslant -\frac{5563}{16384}\right\}.$$

We claim that there exists no points (x^*, z_1^*) in D_i, $i = 1, 2, 3$, for which $q_1(x, z_1) = 0$. As a matter of fact, we show that $q_1(x, z_1) < 0$ for all $(x, z_1) \in D_i$, $i = 1, 2$, and $q_1(x, z_1) > 0$ for all $(x, z_1) \in D_3$.

By calculating the first order partial derivatives of $q_1(x, z_1)$ with respect to x and z_1, respectively, we get

$$\begin{aligned} q_{1x}(x, z_1) = {} & 120x^5 - 350x^4 + 100z_1x^4 + 336x^3 \\ & - 280z_1x^3 + 80{z_1}^2x^3 - 105x^2 + 252z_1x^2 \\ & - 210{z_1}^2x^2 + 60x^2{z_1}^3 - 70z_1x + 168{z_1}^2x \\ & - 140x{z_1}^3 + 40x{z_1}^4 - 35{z_1}^2 \\ & + 84{z_1}^3 - 70{z_1}^4 + 20{z_1}^5, \\ q_{1z_1}(x, z_1) = {} & 20x^5 - 70x^4 + 40z_1x^4 + 84x^3 - 140z_1x^3 \\ & + 60{z_1}^2x^3 - 35x^2 \\ & + 168z_1x^2 - 210{z_1}^2x^2 + 80x^2{z_1}^3 - 70z_1x \\ & + 252{z_1}^2x - 280x{z_1}^3 + 100x{z_1}^4 - 105{z_1}^2 \\ & + 336{z_1}^3 - 350{z_1}^4 + 120{z_1}^5. \end{aligned}$$

Eliminating the variable z_1 by computing the resultant $Q(x)$ of $q_{1x}(x, z_1)$ and $q_{1z_1}(x, z_1)$, we obtain

$$\begin{aligned} Q(x) = {} & -161347200000x^4(2x - 1) \\ & \times (30x^4 - 60x^3 + 24x^2 + 6x + 1) \\ & \times (23040000x^{12} - 138240000x^{11} \\ & + 345600000x^{10} - 460800000x^9 \\ & + 345465600x^8 - 137702400x^7 \\ & + 22569600x^6 - 470400x^5 + 874384x^4 \\ & - 337568x^3 - 196x^2 + 980x + 1225)(x - 1)^4, \end{aligned}$$

which does not vanish in the interval $[\frac{10797}{16384}, \frac{5399}{8192}]$, with the help of Sturm's Theorem. This means that $q_1(x, z_1)$ has no critical points in the interior of D_i, for $i = 1, 2, 3$, such that $q_{1x}(x, z_1) = q_{1z_1}(x, z_1) = 0$ hold. Hence, the extremal values of smooth function $q_1(x, z_1)$ can only be achieved at the boundary of D_i for $i = 1, 2, 3$.

By direct computation, we get the values of the function $q_1(x, z_1)$ at the four vertices of the rectangular region D_1 as follows:

$$\begin{aligned} q_1\left(\frac{10797}{16384}, -\frac{199}{16384}\right) &= -\frac{5870181173924697994698889}{483570327845851669882470 4}, \\ q_1\left(\frac{10797}{16384}, -\frac{99}{8192}\right) &= -\frac{5870715182967575812839165}{4835703278458516698824704}, \\ q_1\left(\frac{5399}{8192}, -\frac{199}{16384}\right) &= -\frac{5870345901975829186245529}{4835703278458516698824704}, \\ q_1\left(\frac{5399}{8192}, -\frac{99}{8192}\right) &= -\frac{917324980849296199663 51}{755578637259143234191 36}. \end{aligned}$$

On one pair of opposite sides of D_1, we obtain for $-\frac{199}{16384} \leqslant z_1 \leqslant -\frac{99}{8192}$ that

$$\begin{aligned} q_1\left(\frac{10797}{16384}, z_1\right) = {} & 20{z_1}^6 - \frac{232735}{4096}{z_1}^5 + \frac{3124304781}{67108864}{z_1}^4 \\ & - \frac{4749788251703}{1099511627776}{z_1}^3 \\ & - \frac{51283463753637291}{18014398509481984}{z_1}^2 \\ & - \frac{5537075581480218309 27}{295147905179352825856}z_1 \\ & - \frac{5978380505324191708518819}{4835703278458516698824704}, \\ q_1\left(\frac{5399}{8192}, z_1\right) = {} & 20{z_1}^6 - \frac{116365}{2048}{z_1}^5 + \frac{781031509}{16777216}{z_1}^4 \\ & - \frac{593574254429}{137438953472}{z_1}^3 \\ & - \frac{3204707399662171}{1125899906842624}{z_1}^2 \\ & - \frac{17302215250776061229}{9223372036854775808}z_1 \\ & - \frac{934146601389399545753 71}{755578637259143234191 36}. \end{aligned}$$

By applying Sturm's Theorem, it can be checked that $q_1\left(\frac{10797}{16384}, z_1\right) \neq 0$ and $q_1\left(\frac{5399}{8192}, z_1\right) \neq 0$. With the same techniques, we can assert on another pair of opposite sides of D_1 that

$$q_1\left(x, -\frac{199}{16384}\right) \neq 0 \quad \textit{and} \quad q_1\left(x, -\frac{99}{8192}\right) \neq 0,$$

when $\frac{10797}{16384} \leqslant x \leqslant \frac{5399}{8192}$.

Summing up the above results, we obtain that $q_1(x, z_1) < 0$ for all (x, z_1) belongs to D_1. Similarly we could confirm that $q_1(x, z_1) < 0$, $q_1(x, z_1) > 0$ for all (x, z_1) in the closed domains D_2, D_3, respectively. Thus, there exists no points $(x^*, z_1^*) \in (0, 1) \times (x_l, 0)$ such that both equations $q_1(x, z_1) = 0$ and $W_{12}(x, z_1) = 0$ hold. Therefore, $W[\ell_{10}, \ell_{11}, \ell_{12}](x) \neq 0$ for all $x \in (0, 1)$.

Finally, from Lemma 2.5, we conclude that the 3-dimensional vector space $E = < J_{10}(h), J_{11}(h), J_{12}(h) >$ is an extended complete Chebyshev space on the interval $\left(0, \frac{1}{140}\right)$. This proves the first part of Theorem 1.2.

Case (b). When $\alpha = \beta = 0$, then system (2) becomes

$$\dot{x} = y, \quad \dot{y} = -x^5(x - 1), \tag{6}$$

with the Hamiltonian function

$$H_2(x, y) = \frac{1}{2}y^2 - \frac{1}{6}x^6 + \frac{1}{7}x^7.$$

We have the continuous family of ovals $L_2(h)$ surrounding the center (1,0) of system (6), where

$$L_2(h) \subset \left\{(x,y) \in \mathbb{R}^2 : H_2(x,y) = h, -\frac{1}{42} < h < 0\right\}.$$

Correspondingly, we consider the complete hyperelliptic integrals of the first kind

$$J_2(h) = \oint_{L_2(h)} \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2}{y} dx, \quad -\frac{1}{42} < h < 0, \quad \alpha_i \in \mathbb{R}, \quad i = 0, 1, 2.$$

As the origin is not a local minimum of $H_2(x, y)$, we translate the center (1,0) of system (6) to the origin by the transformation $x = 1 - u, y = -v$ and still denote the new variables by (x, y) after applying the transformation for convenience. Then system (6) converts into

$$\dot{x} = y, \quad \dot{y} = x(x-1)^5, \tag{7}$$

with the Hamiltonian function $H_2^*(x,y) = A_2(x) + \frac{1}{2}y^2$, which has now a local minimum at the origin. System (7) has a continuous family of ovals $L_2^*(h)$ surrounding the center (0,0), where

$$A_2(x) = \frac{1}{2}x^2 - \frac{5}{3}x^3 + \frac{5}{2}x^4 - 2x^5 + \frac{5}{6}x^6 - \frac{1}{7}x^7,$$

and

$$L_2^*(h) \subset \left\{(x,y) \in \mathbb{R}^2 : H_2^*(x,y) = h, \ 0 < h < \frac{1}{42}\right\}.$$

The Abelian integral $J_2(h)$ can be considered as a linear combination of 3 Abelian integrals $\{J_{20}(h), J_{21}(h), J_{22}(h)\}$, where $J_{2i}(h) = \oint_{L_2^*(h)} x^i y^{2s-1} dx, \ i = 0, 1, 2$ with $s = 0$.

We note that the projection of period annulus $\{L_2^*(h), h \in (0, \frac{1}{42})\}$ on the x-axis is $(-\frac{1}{6}, 1)$ and $xA_2'(x) > 0$ for all $x \in (-\frac{1}{6}, 1) \setminus \{0\}$. Therefore, there exists an invertible function $z_2(x)$ with $-\frac{1}{6} < z_2(x) < 0$ such that $A_2(x) = A_2(z_2(x))$ for $0 < x < 1$; see Fig. 4. But in this case $n = 3$ and $s = 0$, so one of the hypotheses (or assumptions) of Lemma 2.4, i.e. $s > n - 2$ is not fulfilled. However it is possible to overcome this problem using Lemma 2.5, and obtain new Abelian integrals for which the corresponding s is large enough to verify the inequality. Here we have to increase the power s to three so that the condition $s > n - 2$ holds. In order to apply Lemma 2.4 we follow the approach in [6], that consists in multiplying by $A(x) + \frac{1}{2}y^2$ the 1-form and dividing the Abelian integral by h.

Hence, on the oval $L_2^*(h)$ we have

$$\begin{aligned} J_{2i}(h) &= \frac{1}{h}\oint_{L_2^*(h)} \left(A_2(x) + \frac{y^2}{2}\right) x^i y^{-1} dx \\ &= \frac{1}{2h}\left(\oint_{L_2^*(h)} 2x^i A_2(x) y^{-1} dx + \oint_{L_2^*(h)} x^i y dx\right), \\ & i = 0, 1, 2. \end{aligned} \tag{8}$$

Now, we apply Lemma 2.5 with $k = 1$ and $F(x) = 2x^i A_2(x)$ to the first integral above to get

$$\oint_{L_2^*(h)} 2x^i A_2(x) y^{-1} dx = \oint_{L_2^*(h)} G_{2i}(x) y dx,$$

where $G_{2i}(x) = \frac{d}{dx}\left(\frac{2x^i A_2(x)}{A_2'(x)}\right) = \frac{g_{2i}(x)}{21(x-1)^6}$, with

$$\begin{aligned} g_{2i}(x) &= x^i((21i + 21) - (56 + 91\,i)x + (175\,i + 105)x^2 \\ &- (126 + 189\,i)x^3 + (119\,i + 91)x^4 \\ &- (36 + 41\,i)x^5 + (6\,i + 6)x^6). \end{aligned}$$

From (8) we obtain

$$\begin{aligned} J_{2i}(h) &= \frac{1}{2h}\oint_{L_2^*(h)} \left(x^i + G_{2i}(x)\right) y dx \\ &= \frac{1}{4h^2}\oint_{L_2^*(h)} (2A_2(x) + y^2)(x^i + G_{2i}(x)) y dx \\ &= \frac{1}{4h^2}\left(\oint_{L_2^*(h)} 2(x^i + G_{2i}(x))A_2(x) y dx \right. \\ &\left. + \oint_{L_2^*(h)} (x^i + G_{2i}(x)) y^3 dx\right). \end{aligned} \tag{9}$$

Again, we apply Lemma 2.5 with $k = 3$ and $F(x) = 2(x^i + G_{2i}(x))A_2(x)$ to the first integral above to get

$$\oint_{L_2^*(h)} 2(x^i + G_{2i}(x))A_2(x) y dx = \oint_{L_2^*(h)} H_{2i}(x) y^3 dx,$$

where $H_{2i}(x) = \frac{1}{3}\frac{d}{dx}\left(\frac{2(x^i + G_{2i}(x))A_2(x)}{A_2'(x)}\right) = \frac{h_{2i}(x)}{1323(x-1)^{12}}$, with

$$\begin{aligned} h_{2i}(x) &= x^i(882 + 1323\,i + 441\,i^2 \\ &- \left(9996\,i + 4704 + 3822\,i^2\right)x \\ &+ \left(39053\,i + 15631\,i^2 + 1705\right)x^2 \\ &- \left(39788\,i^2 + 46256 + 101626\,i\right)x^3 \\ &+ \left(93492 + 190757\,i + 70021\,i^2\right)x^4 \\ &- \left(140700 + 266042\,i + 89530\,i^2\right)x^5 \\ &+ \left(85085\,i^2 + 278278\,i + 15808\right)x^6 \\ &- \left(132300 + 60424\,i^2 + 217763\,i\right)x^7 \\ &+ \left(125762\,i + 31759\,i^2 + 8145\right)x^8 \\ &- \left(52087\,i + 12026\,i^2 + 35882\right)x^9 \\ &+ \left(3109\,i^2 + 10713 + 14654\,i\right)x^{10} \\ &- \left(1944 + 2511\,i + 492\,i^2\right)x^{11} \\ &+ \left(162 + 36\,i^2 + 198\,i\right)x^{12}). \end{aligned}$$

From (9) we obtain

$$4h^2 J_{2i}(h) = \oint_{L_2^*(h)} f_{2i}(x) y^3 dx \equiv \tilde{J}_{2i}(h), \quad i = 0, 1, 2,$$

where $f_{2i}(x) = x^i + G_{2i}(x) + H_{2i}(x)$. As before, the set $\{\tilde{J}_{20}(h), \tilde{J}_{21}(h), \tilde{J}_{22}(h)\}$ is an extended complete Chebyshev space on $\left(0, \frac{1}{42}\right)$ if and only if $\{J_{20}(h), J_{21}(h), J_{22}(h)\}$ is as well. Now that the condition $s > n - 2$ holds we can use Lemma 2.4. Then, by setting

$$\ell_{2i}(x) = \left(\frac{f_{2i}}{A_2'}\right)(x) - \left(\frac{f_{2i}}{A_2'}\right)(z_2(x)),$$

we need to check that $\{\ell_{20}(x), \ell_{21}(x), \ell_{22}(x)\}$ is a complete Chebyshev space on $x \in (0, 1)$. We claim that this is the case, and we will prove this in the following.

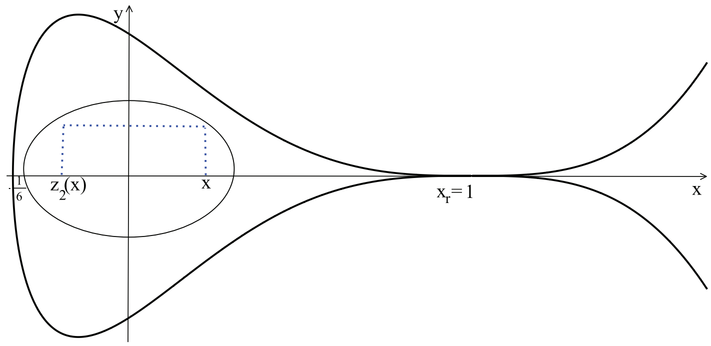

Fig. 4. The involution $z_2(x)$ defined by $A_2(x) = A_2(z_2(x))$.

The involution $z_2 = z_2(x)$ for $x \in (0, 1)$ verifies

$$A_2(x) - A_2(z_2(x)) = -\frac{1}{42}(x - z_2)q_2(x, z_2) = 0,$$

where

$$\begin{aligned} q_2(x, z_2) = & -21 z_2 - 21 x + 6x^6 + 6 z_2{}^6 - 35 x^5 + 70 z_2{}^2 \\ & + 84 x^4 - 35 z_2{}^5 + 84 z_2{}^4 - 105 z_2{}^3 + 70 x^2 \\ & - 105 x^3 + 84 z_2x^3 - 105 z_2x^2 + 70 z_2x \\ & + 84 z_2{}^2x^2 - 105 z_2{}^2x + 6 z_2x^5 - 35 z_2x^4 \\ & + 6 x^3z_2{}^3 - 35 x^2z_2{}^3 + 6 x^2z_2{}^4 + 84 xz_2{}^3 \\ & - 35 xz_2{}^4 + 6 xz_2{}^5 + 6 z_2{}^2x^4 - 35 z_2{}^2x^3. \end{aligned}$$

Using Maple, we find that

$$\begin{aligned} W[\ell_{20}](x) &= \frac{(x - z_2)W_{20}(x, z_2)}{1323(x-1)^{17}x(z_2-1)^{17}z_2}, \\ W[\ell_{20}, \ell_{21}](x) &= \frac{(x - z_2)^3 W_{21}(x, z_2)}{z_2^2(z_2-1)^{34}x^2(x-1)^{34}W_{02}(x, z_2)}, \\ W[\ell_{20}, \ell_{21}, \ell_{22}](x) &= \frac{(x - z_2)^6 W_{22}(x, z_2)}{z_2^3(z_2-1)^{51}x^3(x-1)^{51}W_{02}^3(x, z_2)}, \end{aligned}$$

where $W_{2i}(x, z_2),\ i = 0, 1, 2$ are polynomials in (x, z_2) with long expressions, and

$$\begin{aligned} W_{02}(x, z_2) = & \ 18 z_2{}^2x^3 - 105 z_2{}^2x^2 + 24 x^2z_2{}^3 + 252z_2{}^2x \\ & - 140 xz_2{}^3 + 30 xz_2{}^4 + 12 z_2x^4 \\ & - 70 z_2x^3 + 168 z_2x^2 - 210 z_2x - 175 z_2{}^4 \\ & + 336 z_2{}^3 - 315 z_2{}^2 + 140 z_2 \\ & + 36 z_2{}^5 + 6 x^5 - 35 x^4 + 84 x^3 \\ & - 105 x^2 + 70 x - 21. \end{aligned}$$

The resultant of $W_{02}(x, z_2)$ and $q_2(x, z_2)$ with respect to z_2 is

$$\begin{aligned} p_{02}(x) = & \ 130691232(-35x^4 + 84x^3 - 105x^2 \\ & + 6x^5 - 21 + 70x). \end{aligned}$$

It is easy to see that $p_{02}(x)$ has no zeros in $(0, 1)$. This implies that $W[\ell_{20}, \ell_{21}](x)$ and $W[\ell_{20}, \ell_{21}, \ell_{22}](x)$ are well defined in the domain $-\frac{1}{6} < z_2 < 0 < x < 1$.

In order to determine whether these three Wronskians can have zeros on $(0, 1)$, we shall rely on the symbolic computations by Maple to compute the resultant of $W_{2i}(x, z_2),\ i = 0, 1, 2$ and $q_2(x, z_2)$ with respect to z_2, and then apply Sturm's Theorem to assert nonexistence of zeros of $p_{2i}(x),\ i = 0, 1, 2$ in $(0, 1)$ where $p_{2i}(x)$ are polynomials of high degree in x which will be introduced below.

The resultant with respect to z_2 between $q_2(x, z_2)$ and $W_{20}(x, z_2)$ is

$$R_{20}(q_2, W_{20}, z_2) = (x-1)^{80}p_{20}(x),$$

where $p_{20}(x)$ is a polynomial of degree 94 in x. By applying Sturm's Theorem, we get that $p_{20}(x) \neq 0$ for all $x \in (0, 1)$. Thus, $W_{20}(x, z_2) = 0$ and $q_2(x, z_2) = 0$ have no common roots. This fact implies that $W[\ell_{20}](x) \neq 0$ for all $x \in (0, 1)$.

The resultant with respect to z_2 between $q_2(x, z_2)$ and $W_{21}(x, z_2)$ is

$$R_{21}(q_2, W_{21}, z_2) = (x-1)^{176}p_{21}(x),$$

where $p_{21}(x)$ is a polynomial of degree 196 in x. By applying Sturm's Theorem, we get that $p_{21}(x) \neq 0$ for all $x \in (0, 1)$. Thus, $W_{21}(x, z_2) = 0$ and $q_2(x, z_2) = 0$ have no common roots, and this gives $W[\ell_{20}, \ell_{21}](x) \neq 0$ for all $x \in (0, 1)$.

The resultant with respect to z_2 between $q_2(x, z_2)$ and $W_{22}(x, z_2)$ is

$$R_{22}(q_2, W_{22}, z_2) = (x-1)^{290}p_{221}(x),$$

where $p_{221}(x)$ is a polynomial of degree 304 in x. By applying Sturm's Theorem, we get however that $p_{221}(x)$ has two zeros in the open interval $(0, 1)$. Additionally, by using the function realroot with accuracy $\frac{1}{10000}$ in Maple, we see that the roots are in the closed subintervals $[\frac{7707}{16384}, \frac{1927}{4096}]$ and $[\frac{9611}{16384}, \frac{2403}{4096}]$ of $(0, 1)$.

For further study, we calculate the resultant of $q_2(x, z_2)$ and $W_{22}(x, z_2)$ with respect to x. Doing so we obtain

$$R_{22}(q_2, W_{22}, x) = (z_2-1)^{290}p_{222}(z_2),$$

where $p_{222}(z_2)$ is a polynomial of degree 304 in z_2. By applying Sturm's Theorem, we get however that $p_{222}(z_2)$ has two real roots in $(-\frac{1}{6}, 0)$. By using the function realroot with accuracy $\frac{1}{10000}$ in Maple, we observe that the two roots are in the closed subintervals $[-\frac{497}{8192}, -\frac{993}{16384}]$ and $[-\frac{2635}{16384}, -\frac{1317}{8192}]$ contained in $(-\frac{1}{6}, 0)$.

Therefore, if the equations $q_2(x, z_2) = 0$ and $W_{22}(x, z_2) = 0$ have one common root $(x^*, z_2^*) \in (0, 1) \times (-\frac{1}{6}, 0)$, the point (x^*, z_2^*) must be in the following four domains:

$$D_1 := \left\{(x, z_2) | \frac{7707}{16384} \leqslant x \leqslant \frac{1927}{4096},\ -\frac{497}{8192} \leqslant z_2 \leqslant -\frac{993}{16384}\right\},$$

$$D_2 := \left\{(x, z_2) \mid \frac{7707}{16384} \leqslant x \leqslant \frac{1927}{4096},\ -\frac{2635}{16384} \leqslant z_2 \leqslant -\frac{1317}{8192}\right\},$$

$$D_3 := \left\{(x, z_2) \,|\, \frac{9611}{16384} \leqslant x \leqslant \frac{2403}{4096},\ -\frac{497}{8192} \leqslant z_2 \leqslant -\frac{993}{16384}\right\},$$
$$D_4 := \left\{(x, z_2) \,|\, \frac{9611}{16384} \leqslant x \leqslant \frac{2403}{4096},\ -\frac{2635}{16384} \leqslant z_2 \leqslant -\frac{1317}{8192}\right\}.$$

We claim that there exists no points (x^*, z_2^*) in D_i, $i = 1, 2, 3, 4$ for which $q_2(x, z_2) = 0$. As a matter of fact, we will prove that $q_2(x, z_2) < 0$ for all $(x, z_2) \in D_i$, $i = 1, 2, 3, 4$.

By calculating the first order partial derivatives of $q_2(x, z_2)$ with respect to x and z_2, respectively, we get

$$\begin{aligned} q_{2x}(x, z_2) &= -21 + 36x^5 - 175x^4 + 336x^3 - 315x^2 \\ &\quad + 140x + 18x^2{z_2}^3 - 70x{z_2}^3 + 12x{z_2}^4 \\ &\quad + 84{z_2}^3 - 35{z_2}^4 + 6{z_2}^5 + 24{z_2}^2x^3 \\ &\quad - 105{z_2}^2x^2 + 168{z_2}^2x - 105{z_2}^2 \\ &\quad + 30z_2x^4 - 140z_2x^3 + 252z_2x^2 \\ &\quad - 210z_2x + 70z_2, \\ q_{2z_2}(x, z_2) &= 18{z_2}^2x^3 - 105{z_2}^2x^2 + 24x^2{z_2}^3 + 252{z_2}^2x \\ &\quad - 140x{z_2}^3 + 30x{z_2}^4 \\ &\quad + 12z_2x^4 - 70z_2x^3 + 168z_2x^2 \\ &\quad - 210z_2x - 21 + 6x^5 - 35x^4 + 84x^3 \\ &\quad - 105x^2 + 70x + 36{z_2}^5 - 175{z_2}^4 \\ &\quad + 336{z_2}^3 - 315{z_2}^2 + 140z_2. \end{aligned}$$

Eliminating the variable z_2 by computing the resultant $Q(x)$ of $q_{2x}(x, z_2)$ and $q_{2z_2}(x, z_2)$, we get that

$$\begin{aligned} Q(x) &= 130691232(6x - 1)(x - 1)^{16} \\ &\quad \times (1679616x^8 - 7838208x^7 + 14183424x^6 \\ &\quad - 12083904x^5 + 4443984x^4 \\ &\quad - 335664x^3 + 10944x^2 - 168x + 1), \end{aligned}$$

which does not vanish on the intervals $[\frac{7707}{16384}, \frac{1927}{4096}]$ and $[\frac{9611}{16384}, \frac{2403}{4096}]$, with the help of Sturm's Theorem. This means that $q_2(x, z_2)$ has no critical points in the interior of D_i, for $i = 1, 2, 3, 4$, such that $q_{2x}(x, z_2) = q_{2z_2}(x, z_2) = 0$ hold. Hence, the extremal values of smooth function $q_2(x, z_2)$ can only be achieved at the boundary of D_i for $i = 1, 2, 3, 4$.

By direct computation, we get the values of the function $q_2(x, z_2)$ at the four vertices of the rectangular region D_1 as follows:

$$\begin{aligned} q_2\left(\frac{7707}{16384}, -\frac{497}{8192}\right) &= -\frac{1495583570905374402796 7283}{9671406556917033397649408}, \\ q_2\left(\frac{7707}{16384}, -\frac{993}{16384}\right) &= -\frac{14961354952757344668189279}{9671406556917033397649408}, \\ q_2\left(\frac{1927}{4096}, -\frac{497}{8192}\right) &= -\frac{2336723695932279764984 83}{1511157274518286468382 72}, \\ q_2\left(\frac{1927}{4096}, -\frac{993}{16384}\right) &= -\frac{14960550170937770638001441}{9671406556917033397649408}. \end{aligned}$$

On one pair of opposite sides of D_1, we obtain for $-\frac{497}{8192} \leqslant z_2 \leqslant -\frac{993}{16384}$ that

$$\begin{aligned} q_2\left(\frac{7707}{16384}, z_2\right) &= 6{z_2}^6 - \frac{263599}{8192}{z_2}^5 + \frac{9242731659}{134217728}{z_2}^4 \\ &\quad - \frac{159663708937047}{2199023255552}{z_2}^3 \\ &\quad + \frac{1291487586549656531}{36028797018963968}{z_2}^2 \\ &\quad - \frac{244271718799461580153 5}{590295810358705651712}z_2 \\ &\quad - \frac{18826021367874503982430245}{9671406556917033397649408}, \\ q_2\left(\frac{1927}{4096}, z_2\right) &= 6{z_2}^6 - \frac{65899}{2048}{z_2}^5 + \frac{577655699}{8388608}{z_2}^4 \\ &\quad - \frac{2494629996667}{34359738368}{z_2}^3 \\ &\quad + \frac{5044472181295651}{140737488355328}{z_2}^2 \\ &\quad - \frac{2384977905015173771}{576460752303423488}z_2 \\ &\quad - \frac{459585242296423985671 7}{236118324143482260684 8}. \end{aligned}$$

By applying Sturm's Theorem, it can be checked that $q_2\left(\frac{7707}{16384}, z_2\right) \neq 0$ and $q_2\left(\frac{1927}{4096}, z_2\right) \neq 0$. With the same techniques, we can assert on another pair of opposite sides of D_1 that

$$q_2\left(x, -\frac{497}{8192}\right) \neq 0 \quad \textit{and} \quad q_2\left(x, -\frac{993}{16384}\right) \neq 0,$$

for $\frac{7707}{16384} \leqslant x \leqslant \frac{1927}{4096}$.

Summarizing the above results, we obtain that $q_2(x, z_2) < 0$ for all (x, z_2) in the closed rectangle D_1. Similarly we could confirm $q_2(x, z_2) < 0$ for all (x, z_2) in the closed domains D_i, $i = 2, 3, 4$. Thus, there exists no point $(x^*, {z_2}^*) \in (0, 1) \times (-\frac{1}{6}, 0)$ such that both equations $q_2(x, z_2) = 0$ and $W_{22}(x, z_2) = 0$ hold. Therefore, $W[\ell_{20}, \ell_{21}, \ell_{22}](x) \neq 0$ for all $x \in (0, 1)$.

Finally, we conclude from Lemma 2.4 that the 3-dimensional vector space $E = \langle J_{20}(h), J_{21}(h), J_{22}(h) \rangle$ is an extended complete Chebyshev space on the open interval $\left(0, \frac{1}{42}\right)$. This proves the second part of Theorem 1.2.

4. Proof of Theorem 1.4

When $\alpha = \beta = 1$, then system (2) becomes

$$\dot{x} = y, \quad \dot{y} = -x(x - 1)^5, \tag{10}$$

with the Hamiltonian function

$$H_3(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{5}{3}x^3 - \frac{5}{2}x^4 + 2x^5 - \frac{5}{6}x^6 + \frac{1}{7}x^7.$$

Hence, we have a continuous family of ovals $L_3(h)$ surrounding the center (1,0), where

$$L_3(h) \subset \left\{(x, y) \in \mathbb{R}^2 : H_3(x, y) = h, -\frac{1}{42} < h < 0\right\}.$$

Correspondingly, we consider the complete hyperelliptic integral of the first kind

$$\begin{aligned} J_3(h) &= \oint_{L_3(h)} \frac{\alpha_0 + \alpha_1 x + \alpha_2 x^2}{y} dx, \quad -\frac{1}{42} < h < 0, \\ &\quad \alpha_i \in \mathbb{R}, \quad i = 0, 1, 2. \end{aligned}$$

As the origin is not a local minimum of $H_3(x, y)$, we translate the center (1,0) of system (10) to the origin by the transformation $x = 1 - u, y = -v$ and still denote the new variables by (x, y) after applying the transformation, for convenience. Then system (10) turns into

$$\dot{x} = y, \quad \dot{y} = x^5(x - 1), \tag{11}$$

with the Hamiltonian function $H_3^*(x,y)=A_3(x)+\frac{1}{2}y^2$, which has now a local minimum at the origin and a continuous family of ovals $L_3^*(h)$ surrounding the center (0,0), where

$$A_3(x)=\frac{1}{6}x^6-\frac{1}{7}x^7,$$

and

$$L_3^*(h)\subset\left\{(x,y)\in\mathbb{R}^2:\ H_3^*(x,y)=h,\ 0<h<\frac{1}{42}\right\}.$$

We note that the projection of the period annulus $\{L_3^*(h), h\in\left(0,\frac{1}{42}\right)\}$ on the x-axis is $(-0.6703320476, 1)$ and $xA_3'(x)>0$ for all $x\in(-0.6703320476,1)\setminus\{0\}$. Therefore, there exists an invertible function $z_3(x)$ with $-0.6703320476<z_3(x)<0$ such that $A_3(x)=A_3(z_3(x))$ for $0<x<1$. First, we will show that $(J_{30}(h),J_{31}(h))$ and $(J_{30}(h),J_{32}(h))$ are two Chebyshev systems with accuracy 1 on $\left(0,\frac{1}{42}\right)$. But in this case $n=2$ and $s=0$, so one of the hypotheses of Lemma 2.4, i.e. $s>n+k-2$ does not hold even for $k=1$. However, it is possible to overcome this problem, using Lemma 2.5, and obtain new Abelian integrals for which the corresponding s is large enough to verify the inequality. Here, we have to increase the power s to three so that the condition $s>n+k-2$ holds. In order to apply Lemma 2.4 we follow the approach in [6], that consists in multiplying by $A(x)+\frac{1}{2}y^2$ the 1-form and dividing the Abelian integral by h. Hence, on the oval $L_3^*(h)$ we have

$$\begin{aligned}J_{3i}(h) &= \frac{1}{h}\oint_{L_3^*(h)}\left(A_3(x)+\frac{y^2}{2}\right)x^iy^{-1}dx\\ &= \frac{1}{2h}\left(\oint_{L_3^*(h)}2x^iA_3(x)y^{-1}dx+\oint_{L_3^*(h)}x^iydx\right),\\ &\quad i=0,1,2.\end{aligned}\tag{12}$$

Now, we apply Lemma 2.5 with $k=1$ and $F(x)=2x^iA_3(x)$ to the first integral above to get

$$\oint_{L_3^*(h)}2x^iA_3(x)y^{-1}dx=\oint_{L_3^*(h)}G_{3i}(x)ydx,$$

where $G_{3i}(x)=\frac{d}{dx}\left(\frac{2x^iA_3(x)}{A_3'(x)}\right)=\frac{-g_{3i}(x)}{21(x-1)^2}$, with

$$g_{3i}(x)=-6x^{i+2}i+13x^{i+1}i-7x^ii-6x^{i+2}+12x^{i+1}-7x^i.$$

From (12) we have

$$\begin{aligned}J_{3i}(h) &= \frac{1}{2h}\oint_{L_3^*(h)}\left(x^i+G_{3i}(x)\right)ydx\\ &= \frac{1}{4h^2}\oint_{L_3^*(h)}(2A_3(x)+y^2)(x^i+G_{3i}(x))ydx\\ &= \frac{1}{4h^2}\left(\oint_{L_3^*(h)}2(x^i+G_{3i}(x))A_3(x)ydx\right.\\ &\quad\left.+\oint_{L_3^*(h)}(x^i+G_{3i}(x))y^3dx\right).\end{aligned}\tag{13}$$

Again, we apply Lemma 2.5 with $k=3$ and $F(x)=2(x^i+G_{3i}(x))A_3(x)$ to the first integral above to get

$$\oint_{L_3^*(h)}2(x^i+G_{3i}(x))A_3(x)ydx=\oint_{L_3^*(h)}H_{3i}(x)y^3dx,$$

where $H_{3i}(x)=\frac{1}{3}\frac{d}{dx}\left(\frac{2(x^i+G_{3i}(x))A_3(x)}{A_3'(x)}\right)=\frac{-h_{3i}(x)}{1323(x-1)^4}$, with

$$\begin{aligned}h_{3i}(x) &= 700x^{i+1}-993x^{i+2}+910x^{i+1}i-245x^ii\\ &\quad-1286x^{i+2}i-196x^i+648x^{i+3}\\ &\quad-198x^{i+4}i+819x^{i+3}i-36x^{i+4}i^2+156x^{i+3}i^2\\ &\quad-49x^ii^2-253x^{i+2}i^2\\ &\quad+182x^{i+1}i^2-162x^{i+4}.\end{aligned}$$

From (13) we obtain

$$4h^2J_{3i}(h)=\oint_{L_3^*(h)}f_{3i}(x)y^3dx\equiv\tilde{J}_{3i}(h),\quad i=0,1,2,$$

where $f_{3i}(x)=x^i+G_{3i}(x)+H_{3i}(x)$.

Now, we will prove that the 2-dimensional linear spaces of $\{\tilde{J}_{30}(h),\tilde{J}_{31}(h)\}$ and $\{\tilde{J}_{30}(h),\tilde{J}_{32}(h)\}$ are Chebyshev spaces with accuracy one on the interval $\left(0,\frac{1}{42}\right)$ by the conclusion (ii) of Lemma 2.4. Due to Lemma 2.4, by setting

$$\ell_{3i}(x)=\left(\frac{f_{3i}}{A_3'}\right)(x)-\left(\frac{f_{3i}}{A_3'}\right)(z_3(x)),$$

we only need to check that $(\ell_{30}(x),\ell_{31}(x))$ and $(\ell_{30}(x),\ell_{32}(x))$ are two Chebyshev systems with accuracy one on $x\in(0,1)$. We claim that this is the case. To prove the claim, we will follow exactly the same approach as in the cases (a) and (b) of Theorem 1.3.

The involution $z_3=z_3(x)$ for $x\in(0,1)$ verifies that

$$A_3(x)-A_3(z_3(x))=-\frac{1}{42}(x-z_3)q_3(x,z_3)=0,$$

where

$$\begin{aligned}q_3(x,z_3) &= 6x^6-7x^5+6z_3x^5-7z_3x^4+6{z_3}^2x^4\\ &\quad-7{z_3}^2x^3+6x^3{z_3}^3-7x^2{z_3}^3\\ &\quad+6x^2{z_3}^4-7x{z_3}^4+6x{z_3}^5-7{z_3}^5+6{z_3}^6.\end{aligned}$$

Hence, $z_3(x)$ is implicitly determined by the equation $q_3(x,z_3)=0$, and its derivative with respect to x is given by $\frac{dz_3}{dx}=-\frac{q_{3x}(x,z_3)}{q_{3z_3}(x,z_3)}$.

Using Maple, we find that

$$\begin{aligned}W[\ell_{30}](x) &= \frac{(x-z_3)W_{30}(x,z_3)}{1323(x-1)^5x^5(z_3-1)^5z_3^5},\\ W[\ell_{30},\ell_{31}](x) &= -\frac{5(x-z_3)^3W_{31}(x,z_3)}{1750329z_3^{10}(z_3-1)^9x^{10}(x-1)^9W_{03}(x,z_3)},\\ W[\ell_{30},\ell_{32}](x) &= -\frac{(x-z_3)^3W_{32}(x,z_3)}{1750329z_3^{9}(z_3-1)^9x^{9}(x-1)^9W_{03}(x,z_3)},\end{aligned}$$

where $W_{3i}(x,z_3)$, $i=0,1,2$ are polynomials in (x,z_3) with long expressions, and

$$\begin{aligned}W_{03}(x,z_3) &= 6x^5-7x^4+12z_3x^4-14z_3x^3\\ &\quad+18{z_3}^2x^3-21{z_3}^2x^2+24x^2{z_3}^3-28x{z_3}^3\\ &\quad+30x{z_3}^4-35{z_3}^4+36{z_3}^5.\end{aligned}$$

The resultant with respect to z_3 between $W_{03}(x,z_3)$ and $q_3(x,z_3)$ is

$$\begin{aligned}p_{03}(x) &= 130691232x^{20}\left(6x^5+5x^4+4x^3+3x^2+2x+1\right)\\ &\quad\times(6x-7)^5.\end{aligned}$$

It is easy to see that $p_{03}(x)$ has no zeros in $(0,1)$. This implies that $W[\ell_{30},\ell_{31}](x)$ and $W[\ell_{30},\ell_{32}](x)$ are well defined in the domain $-0.6703320476<z_3<0<x<1$.

The resultant with respect to z_3 between $q_3(x, z_3)$ and $W_{30}(x, z_3)$ is

$$R_{30}(q_3, W_{30}, z_3) = x^{20}(x-1)^4 p_{30}(x),$$

where $p_{30}(x)$ is a polynomial of degree 54 in x. By applying Sturm's Theorem, we see that $p_{30}(x) \neq 0$ for all $x \in (0, 1)$. Thus, $W_{30}(x, z_3) = 0$ and $q_3(x, z_3) = 0$ have no common roots. This fact implies that $W[\ell_{30}](x) \neq 0$ for all $x \in (0, 1)$.

The resultant with respect to z_3 between $q_3(x, z_3)$ and $W_{31}(x, z_3)$ is

$$R_{31}(q_3, W_{31}, z_3) = x^{56}(x-1)^6 p_{311}(x),$$

where $p_{311}(x)$ is a polynomial of degree 106 in x. By applying Sturm's Theorem, we find however that $p_{311}(x)$ has two zeros in the open interval $(0, 1)$. In addition, by using the function realroot with accuracy $\frac{1}{10000}$ in Maple, we observe that the roots are in the closed subintervals $[\frac{6531}{8192}, \frac{13063}{16384}]$ and $[\frac{961}{1024}, \frac{15377}{16384}]$ of $(0, 1)$.

For further study, we calculate the resultant of $q_3(x, z_3)$ and $W_{31}(x, z_3)$ with respect to x. Doing so we get

$$R_{31}(q_3, W_{31}, x) = z_3^{56}(z_3-1)^6 p_{312}(z_3),$$

where $p_{312}(z_3)$ is a polynomial of degree 106 in z_3. By applying Sturm's Theorem, we see that $p_{312}(z_3)$ has one root in $(-0.6703320476, 0)$. By using the function realroot with accuracy $\frac{1}{10000}$ in Maple, we claim the one root is in the closed subinterval $[-\frac{10869}{16384}, -\frac{2717}{4096}]$ contained in $(-0.6703320476, 0)$.

Therefore, if the equations $q_3(x, z_3) = 0$ and $W_{31}(x, z_3) = 0$ have one common root $(x^*, z_3^*) \in (0, 1) \times (-0.6703320476, 0)$, the point (x^*, z_3^*) must be in the following two domains:

$$D_1 := \left\{(x, z_3) | \frac{6531}{8192} \leqslant x \leqslant \frac{13063}{16384},\ -\frac{10869}{16384} \leqslant z_3 \leqslant -\frac{2717}{4096}\right\},$$
$$D_2 := \left\{(x, z_3) \mid \frac{961}{1024} \leqslant x \leqslant \frac{15377}{16384},\ -\frac{10869}{16384} \leqslant z_3 \leqslant -\frac{2717}{4096}\right\}.$$

We claim that there exists no points (x^*, z_3^*) in D_1 for which $q_3(x, z_3) = 0$. As a matter of fact, we will prove that $q_3(x, z_3) < 0$ for all $(x, z_3) \in D_1$.

By calculating the first order partial derivatives of $q_3(x, z_3)$ with respect to x and z_3, respectively, we get

$$\begin{aligned} q_{3x}(x, z_3) &= 6x^5 - 7x^4 + 12z_3x^4 - 14z_3x^3 \\ &\quad + 18{z_3}^2x^3 - 21{z_3}^2x^2 + 24x^2{z_3}^3 \\ &\quad - 28x{z_3}^3 + 30x{z_3}^4 - 35{z_3}^4 + 36{z_3}^5, \\ q_{3z_3}(x, z_3) &= 36x^5 - 35x^4 + 30z_3x^4 - 28z_3x^3 \\ &\quad + 24{z_3}^2x^3 - 21{z_3}^2x^2 + 18x^2{z_3}^3 \\ &\quad - 14x{z_3}^3 + 12x{z_3}^4 - 7{z_3}^4 + 6{z_3}^5. \end{aligned}$$

Eliminating the variable z_3 by computing the resultant $Q(x)$ of $q_{3x}(x, z_3)$ and $q_{3z_3}(x, z_3)$, we obtain

$$\begin{aligned} Q(x) &= 130691232x^{16}(6x-5)(1679616x^8 - 5598720x^7 \\ &\quad + 6345216x^6 - 2472768x^5 + 11664x^4 \\ &\quad + 9072x^3 + 7056x^2 - 41160x + 60025), \end{aligned}$$

which does not vanish in both intervals $[\frac{6531}{8192}, \frac{13063}{16384}]$ and $[\frac{961}{1024}, \frac{15377}{16384}]$, with the help of Sturm's Theorem. This means that the function $q_3(x, z_3)$ has no critical points in D_i, $i = 1, 2$, such that $q_{3x}(x, z_3) = q_{3z_3}(x, z_3) = 0$ hold. Hence, the extremal values of the smooth function $q_3(x, z_3)$ can only be achieved at the boundary of D_i for $i = 1, 2$.

By direct computation, we get the values of the function $q_3(x, z_3)$ at the four vertices of the rectangular region D_1 as follows:

$$\begin{aligned} q_3\left(\frac{6531}{8192}, -\frac{10869}{16384}\right) &= \frac{242854535979551435985918 9}{967140655691703339764940 8}, \\ q_3\left(\frac{6531}{8192}, -\frac{2717}{4096}\right) &= \frac{378909372417712468617 49}{151115727451828646838272}, \\ q_3\left(\frac{13063}{16384}, -\frac{10869}{16384}\right) &= \frac{2427335458500051204680473}{9671406556917033397649408}, \\ q_3\left(\frac{13063}{16384}, -\frac{2717}{4096}\right) &= \frac{2423810178934197119983647}{9671406556917033397649408}. \end{aligned}$$

On one pair of opposite sides of D_1, we obtain for $-\frac{10869}{16384} \leqslant z_3 \leqslant -\frac{2717}{4096}$ that

$$\begin{aligned} q_3\left(\frac{6531}{8192}, z_3\right) &= 6{z_3}^6 - \frac{9079}{4096}{z_3}^5 - \frac{59294949}{33554432}{z_3}^4 \\ &\quad - \frac{387255311919}{274877906944}{z_3}^3 \\ &\quad - \frac{2529164442142989}{2251799813685248}{z_3}^2 \\ &\quad - \frac{16517972971635861159}{18446744073709551616}z_3 \\ &\quad - \frac{107878881477753809229429}{151115727451828646838272}, \\ q_3\left(\frac{13063}{16384}, z_3\right) &= 6{z_3}^6 - \frac{18155}{8192}{z_3}^5 - \frac{237158765}{134217728}{z_3}^4 \\ &\quad - \frac{3098004947195}{2199023255552}{z_3}^3 \\ &\quad - \frac{40469238625208285}{36028797018963968}{z_3}^2 \\ &\quad - \frac{52864966416109582695 5}{590295810358705651712}z_3 \\ &\quad - \frac{6905750562936394787513165}{9671406556917033397649408}. \end{aligned}$$

By applying Sturm's Theorem, it can be checked that $q_3\left(\frac{6531}{8192}, z_3\right) \neq 0$ and $q_3\left(\frac{13063}{16384}, z_3\right) \neq 0$. With the same techniques, we can assert on another pair of opposite sides of D_1 that

$$q_3\left(x, -\frac{10869}{16384}\right) \neq 0 \quad \textit{and} \quad q_3\left(x, -\frac{2717}{4096}\right) \neq 0,$$

when $\frac{6531}{8192} \leqslant x \leqslant \frac{13063}{16384}$.

Summarizing the above results, we obtain that $q_3(x, z_3) > 0$ for all (x, z_3) belongs to D_1. Therefore, there exists a unique $x^* \in (0, 1)$, with $\frac{961}{1024} \leq x^* \leq \frac{15377}{16384}$, so that $W[\ell_{30}, \ell_{31}](x^*) = 0$. We will now show that x^* is a simple root. Let us denote $W[\ell_{30}, \ell_{31}](x)$ by $W_2(x, z_3(x))$ and calculate its derivative, that is

$$\begin{aligned} \frac{dW_2}{dx} &= \frac{\partial W_2}{\partial x} + \frac{\partial W_2}{\partial z_3} \times \frac{dz_3}{dx} \\ &= -\frac{5(x-z_3)^2 w_4(x, z_3)}{1750329{z_3}^{10}(z_3-1)^9 x^{11}(x-1)^{10} W_{03}^2(x, z_3)}, \end{aligned}$$

where $w_4(x, z_3)$ is a polynomial of degree 35. The resultant with respect to z_3 between $q_3(x, z_3)$ and $w_4(x, z_3)$ is

$$R_4(q_3, w_4, z_3) = x^{80}(x-1)^7 p_4(x),$$

where $p_4(x)$ is a polynomial of degree 123 in x. By applying Sturm's Theorem, we find that $p_4(x)$ has no zeros in $[\frac{961}{1024}, \frac{15377}{16384}]$. Therefore, $W[\ell_{30}, \ell_{31}](x)$ has exactly one simple root in the interval $(0, 1)$.

The resultant with respect to z_3 between $q_3(x, z_3)$ and $W_{32}(x, z_3)$ is

$$R_{32}(q_3, W_{32}, z_3) = x^{56}(x-1)^6 p_{32}(x),$$

where $p_{32}(x)$ is a polynomial of degree 104 in x. By applying Sturm's Theorem, we find that $p_{32}(x)$ has two zeros in the open interval $(0, 1)$. By using the same arguments as above and according to the above analysis, we conclude that the equations $q_3(x, z_3) = 0$ and $W_{32}(x, z_3) = 0$ have one common root $(x^*, z_3^*) \in (0, 1) \times (-0.6703320476, 0)$, with

$$x^* \in \left[\frac{14361}{16384}, \frac{7181}{8192}\right], \quad z_3^* \in \left[-\frac{10583}{16384}, -\frac{5291}{8192}\right].$$

Therefore, $W[\ell_{30}, \ell_{32}](x)$ has exactly one simple root in the interval $(0, 1)$.

By the conclusion (ii) of Lemma 2.4, we can say that the 2-dimensional subspaces $E_1 =< J_{30}(h), J_{31}(h) >$ and $E_2 =< J_{30}(h), J_{32}(h) >$ are Chebyshev with accuracy one on the interval $\left(0, \frac{1}{42}\right)$. It follows that any linear combination of $\{J_{30}(h), J_{31}(h)\}$ or $\{J_{30}(h), J_{32}(h)\}$ has at most two isolated zeros (counting multiplicity) in the interval $\left(0, \frac{1}{42}\right)$. We will consider this as the first conclusion (the first part) of Theorem 1.4.

Next, we will prove that the upper bound two (mentioned above) can be reduced to one. This will be done by computing asymptotic developments of Abelian integrals at the end points $h = 0$ and $h = \frac{1}{42}$, and making a geometric analysis.

We start by defining the following integrals:

$$I_{30}(h) := \oint_{L_3^*(h)} y dx, \qquad I_{31}(h) := \oint_{L_3^*(h)} xy dx,$$
$$I_{32}(h) := \oint_{L_3^*(h)} x^2 y dx.$$

When $0 < h < \frac{1}{42}$, it is obvious that

$$J_{30}(h) = \frac{dI_{30}(h)}{dh}, \; J_{31}(h) = \frac{dI_{31}(h)}{dh}, \; J_{32}(h) = \frac{dI_{32}(h)}{dh}.$$

When $0 < h \ll 1$, the projection of the period annulus $\{L_3^*(h)\}_h$ on the x-axis is the interval $(x_1(h), x_2(h))$ with

$$x_2(h) = g^{-1}(\sqrt[6]{h}) = \sqrt[6]{6}\sqrt[6]{h} + \frac{1}{7}\sqrt[3]{6}\sqrt[3]{h} + \frac{9}{98}\sqrt{6}\sqrt{h} + \frac{80}{1029} 6^{\frac{2}{3}} h^{\frac{2}{3}} + \frac{4301}{57624} 6^{\frac{5}{6}} h^{\frac{5}{6}} + O(h) > 0,$$

$$x_1(h) = g^{-1}(-\sqrt[6]{h}) = -\sqrt[6]{6}\sqrt[6]{h} + \frac{1}{7}\sqrt[3]{6}\sqrt[3]{h} - \frac{9}{98}\sqrt{6}\sqrt{h} + \frac{80}{1029} 6^{\frac{2}{3}} h^{\frac{2}{3}} - \frac{4301}{57624} 6^{\frac{5}{6}} h^{\frac{5}{6}} + O(h) < 0,$$

where $g(x) = sign(x)\sqrt[6]{A_3(x)}$.

Now we compute the asymptotic expansions of $I_{30}(h)$, $I_{31}(h), I_{32}(h)$, for $0 < h \ll 1$, as follows:

$$I_{30}(h) = \oint_{L_h^*} y dx = 2\int_{x_1(h)}^{x_2(h)} \sqrt{2h - 2\left(\frac{1}{6}x^6 - \frac{1}{7}x^7\right)} dx$$
$$= 4(2^{\frac{2}{3}}))(3^{\frac{1}{6}})h^{\frac{2}{3}}\Bigg(1 - \frac{1}{14} - \frac{1}{104} + \dots + \frac{\left(\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-n+1\right)\right)}{n!}\frac{(-1)^n}{6n+1}\Bigg)$$
$$+ \frac{108}{49} h\sqrt{3}\Bigg(\frac{1}{3} - \frac{1}{18} - \frac{1}{120} + \dots + \frac{\left(\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-n+1\right)\right)}{n!}\frac{(-1)^n}{6n+3}\Bigg) + \dots + O\left(h^2\right)$$
$$= \frac{2}{3} 2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{1}{6}, \frac{2}{3}\right)h^{\frac{2}{3}} + \frac{18}{49}\sqrt{3}B\left(\frac{3}{6}, \frac{2}{3}\right)h + \frac{21505}{43218}\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{2}{3}\right)h^{\frac{4}{3}} + O(h^{\frac{5}{3}}),$$

$$I_{31}(h) = 2\int_{x_1(h)}^{x_2(h)} x\sqrt{2h - 2(\frac{1}{6}x^6 - \frac{1}{7}x^7)}dx$$
$$= \frac{2\pi\sqrt{3}}{7}h + \frac{1870}{3087}\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{3}{2}\right)h^{4/3} + \frac{38285}{14406} 2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{7}{6}, \frac{3}{2}\right)h^{5/3} + O\left(h^2\right),$$

$$I_{32}(h) = 2\int_{x_1(h)}^{x_2(h)} x^2\sqrt{2h - 2(\frac{1}{6}x^6 - \frac{1}{7}x^7)}dx$$
$$= \frac{2\pi}{\sqrt{3}}h + \frac{110}{147}\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{3}{2}\right)h^{4/3} + \frac{6175}{2058} 2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{6}{7}, \frac{3}{2}\right)h^{5/3} + O\left(h^2\right).$$

Therefore, when $0 < h \ll 1$, we can write

$$J_{30}(h) = I'_{30}(h) = \frac{4}{9}2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{1}{6}, \frac{2}{3}\right)\frac{1}{\sqrt[3]{h}} + \frac{18}{49}\sqrt{3}B\left(\frac{1}{2}, \frac{2}{3}\right) + \frac{43010}{64827}\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{2}{3}\right)\sqrt[3]{h} + O(h^{\frac{2}{3}}),$$

$$J_{31}(h) = I'_{31}(h) = \frac{2\pi\sqrt{3}}{7} + \frac{7480}{9261}\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{3}{2}\right)\sqrt[3]{h} + \frac{191425}{43218} 2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{7}{6}, \frac{3}{2}\right)h^{\frac{2}{3}} + O(h),$$

$$J_{32}(h) = I'_{32}(h) = \frac{2\pi}{\sqrt{3}} + \frac{440}{441}\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{3}{2}\right)\sqrt[3]{h} + \frac{30875}{6174} 2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{6}{7}, \frac{3}{2}\right)h^{\frac{2}{3}} + O(h).$$

The first derivatives of $J_{3k}(h)$, $k = 0, 1, 2$, are

$$J'_{30}(h) = -\frac{4}{27}\frac{2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{1}{6}, \frac{2}{3}\right)}{h^{4/3}} + \frac{43010}{194481}\frac{\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{2}{3}\right)}{h^{\frac{2}{3}}} + \frac{7082725}{2722734}\frac{2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{7}{6}, \frac{2}{3}\right)}{\sqrt[3]{h}} + O(1),$$

$$J'_{31}(h) = \frac{7480}{27783}\frac{\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{3}{2}\right)}{h^{\frac{2}{3}}} + \frac{191425}{64827}\frac{2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{7}{6}, \frac{3}{2}\right)}{\sqrt[3]{h}} + O(1),$$

$$J'_{32}(h) = \frac{440}{1323}\frac{\sqrt[3]{2}3^{\frac{5}{6}}B\left(\frac{5}{6}, \frac{3}{2}\right)}{h^{\frac{2}{3}}} + \frac{30875}{9261}\frac{2^{\frac{2}{3}}\sqrt[6]{3}B\left(\frac{6}{7}, \frac{3}{2}\right)}{\sqrt[3]{h}} + O(1).$$

Here, $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{1}{\alpha} + \sum_{n=1}^{+\infty} \frac{(-1)^n(\beta-1)(\beta-2)\dots(\beta-n)}{n!(\alpha+n)}$ is the Beta-function.

On the other hand, we consider the asymptotic expansion of $I_{30}(h), I_{31}(h), I_{32}(h)$ near $h = \frac{1}{42}$. We make the transformation $X = 1-x, Y = -y$ on system (11) in order that the results in [8] can be used. Then system (11) can be written as

$$\dot{X} = Y, \quad \dot{Y} = -X(X-1)^5, \tag{14}$$

which has the Hamiltonian function

$$H_3(X,Y) = \frac{1}{2}Y^2 - \frac{1}{2}X^2 + \frac{5}{3}X^3 - \frac{5}{2}X^4 + 2X^5 - \frac{5}{6}X^6 + \frac{1}{7}X^7,$$

with the continuous family of ovals $\tilde{L}_h$ surrounding the nilpotent center $(1, 0)$, where

$$\tilde{L}_h \subset \left\{(X,Y) \in \mathbb{R}^2 : H_3(X,Y) = h, -\frac{1}{42} < h < 0\right\}.$$

Accordingly,

$$\begin{aligned}
I_{30}(h) &= \oint_{L_h^*} ydx = \oint_{\tilde{L}_h} YdX =: \tilde{I}_0(h),\\
I_{31}(h) &= \oint_{L_h^*} xydx = \oint_{\tilde{L}_h} YdX - \oint_{\tilde{L}_h} XYdX =: \tilde{I}_0(h) - \tilde{I}_1(h),\\
I_{32}(h) &= \oint_{L_h^*} x^2ydx = \oint_{\tilde{L}_h} YdX - 2\oint_{\tilde{L}_h} XYdX + \oint_{\tilde{L}_h} X^2YdX\\
&=: \tilde{I}_0(h) - 2\tilde{I}_1(h) + \tilde{I}_2(h).
\end{aligned}$$

Following the results in [8], we can give the asymptotic expansion of $\tilde{I}_0(h), \tilde{I}_1(h)$, and $\tilde{I}_2(h)$ as follows:

$$\begin{aligned}
\tilde{I}_0(h) &= 0.628538 - h\ln(-h) - b_1h - b_2h^2\ln(-h) + O(h^2),\\
\tilde{I}_1(h) &= 0.5581373416 + 15.27202674\,h + O(h^2),\\
\tilde{I}_2(h) &= 0.6115347006 + 17.63994228\,h + O(h^2).
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
I_{30}(h) &= 0.628538 - h\ln(-h) - b_1h - b_2h^2\ln(-h) + O(h^2),\\
I_{31}(h) &= 0.0704003 - h\ln(-h) - (b_1 + 15.2758)h\\
&\quad - b_2h^2\ln(-h) + O(h^2),\\
I_{32}(h) &= 0.123798 - h\ln(-h) - (b_1 + 12.9117)h\\
&\quad - b_2h^2\ln(-h) + O(h^2).
\end{aligned}$$

Consequently,

$$\begin{aligned}
J_{30}(h) &= -\ln(-h) - 1 - b_1 - 2b_2h\ln(-h) - b_2h + O(h),\\
J_{31}(h) &= -\ln(-h) - 16.2758 - b_1 - 2b_2h\ln(-h)\\
&\quad - b_2h + O(h),\\
J_{32}(h) &= -\ln(-h) - 13.9117 - b_1 - 2b_2h\ln(-h)\\
&\quad - b_2h + O(h).
\end{aligned}$$

This yields that

$$\begin{aligned}
J'_{30}(h) &= -h^{-1} - 2b_2\ln(-h) + O(1),\\
J'_{31}(h) &= -h^{-1} - 2b_2\ln(-h) + O(1),\\
J'_{32}(h) &= -h^{-1} - 2b_2\ln(-h) + O(1).
\end{aligned}$$

Let us now define the following functions

$$P_3(h) := \frac{J_{31}(h)}{J_{30}(h)}, \quad Q_3(h) := \frac{J_{32}(h)}{J_{30}(h)}, \quad R_3(h) := \frac{J_{31}(h)}{J_{32}(h)}, \quad h \in \left(0, \frac{1}{42}\right).$$

Then we have the following conclusion:

Lemma 4.1. *The following properties hold:*

(i) $P_3(0^+) = Q_3(0^+) = 0,\ R_3(0^+) = \frac{3}{7}$;

(ii) $P_3\left(\frac{1}{42} - 0\right) = Q_3\left(\frac{1}{42} - 0\right) = R_3\left(\frac{1}{42} - 0\right) = 1$;

(iii) $P'_3(0^+) = Q'_3(0^+) = R'_3(0^+) = +\infty$;

(iv) $P'_3\left(\frac{1}{42} - 0\right) = Q'_3\left(\frac{1}{42} - 0\right) = R'_3\left(\frac{1}{42} - 0\right) = +\infty$.

Proof. The proof is straightforward and follows directly from the asymptotic expansions given above. □

It follows from Lemma 4.1 and the first part of Theorem 1.4 that the functions $P_3(h)$ and $Q_3(h)$ are monotonically increasing on the interval $(0, 1/42)$ with the end points $(0, 0)$ and $(1/42, 1)$. This can be seen as follows. By contradiction, suppose that $P'_3(h)$ not be positive on $(0, 1/42)$. Then, according to conclusions (iii) and (iv) in Lemma 4.1, $P'_3(h)$ will have at least two roots in the interval $(0, 1/42)$. Moreover, the number of zeros of $P'_3(h)$ will be even. Thus, $\alpha_0 + \alpha_1 P_3(h) = 0$ can have at least three roots for some appropriate α_0 and α_1. This leads that $J_3(h) = \alpha_0 J_{30}(h) + \alpha_1 J_{31}(h)$ has at least three zeros for some well chosen α_0 and α_1. This fact contradicts the first conclusion of Theorem 1.4. Therefore, $P'_3(h) > 0$ for all $h \in (0, 1/42)$. The proof for $Q_3(h)$ goes along the same lines.

Finally, we have proved that the exact upper bound on the number of isolated zeros of $J_3(h) = \alpha_0 J_{30}(h) + \alpha_1 J_{31}(h)$ and $J_3(h) = \alpha_0 J_{30}(h) + \alpha_2 J_{32}(h)$ in $(0, 1/42)$ is one for all real α_0, α_1 and α_2. This gives Theorem 1.4.

5. Some additional results

In conclusion, we intend to prove two another results on the monotonicity of the ratio of two Abelian integrals, defined by

$$P(h) := \frac{I_{31}(h)}{I_{30}(h)}, \quad Q(h) := \frac{I_{32}(h)}{I_{30}(h)}.$$

We have the following result.

Lemma 5.1.

(i) $Q'(h) > 0$ *for all* $h \in \left(0, \frac{1}{42}\right)$;

(ii) $P'(h) > 0$ *for all* $h \in \left(0, \frac{1}{42}\right)$.

Proof. We know that the equation of L_h is

$$\frac{1}{2}y^2 - \frac{1}{7}x^7 + \frac{1}{6}x^6 = h, \quad h \in (0, \frac{1}{42}).$$

Assume that L_h has two intersection points with x-axis, denoted respectively by $(\mu(h), 0)$ and $(\nu(h), 0)$ with $\alpha < \mu(h) < 0 < \nu(h) < 1$ where $\alpha \approx -0.67$.

(i) According to Theorem 2 of [2], to prove $Q'(h) > 0$, we will use the criterion function

$$\zeta(x) = \frac{f_2(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_2(\tilde{x})\sqrt{\phi(x)}\Phi'(x)}{f_1(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_1(\tilde{x})\sqrt{\phi(x)}\Phi'(x)},$$

where $\tilde{x} = \tilde{x}(x)$ is defined by $\Phi(x) = \Phi(\tilde{x})$ for $\alpha < x < 0 < \tilde{x} < 1$.

In our case, since

$$Q(h) = \frac{I_2(h)}{I_0(h)} = \frac{\oint_{L_h} x^2 y dx}{\oint_{L_h} 1 y dx}, \text{ and}$$

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{6}x^6 - \frac{1}{7}x^7,$$

we take

$$f_2(x) = x^2, f_1(x) = 1, \phi(x) = \frac{1}{2}, \Phi(x) = \frac{1}{6}x^6 - \frac{1}{7}x^7,$$

then

$$\zeta(x) = \frac{x^2\tilde{x}^5(1-\tilde{x}) - \tilde{x}^2 x^5(1-x)}{\tilde{x}^5(1-\tilde{x}) - x^5(1-x)}.$$

Let $\psi(x) = x^5(1-x)$, then the criterion function $\zeta(x)$ is simplified as

$$\zeta(x) = \frac{x^2\psi(\tilde{x}) - \tilde{x}^2\psi(x)}{\psi(\tilde{x}) - \psi(x)}.$$

Differentiating $\zeta(x)$ with respect to x, we have

$$\zeta'(x) = \zeta_x + \zeta_{\tilde{x}}\frac{d\tilde{x}}{dx},$$

where

$$\zeta_x = \frac{\psi(\tilde{x})(2x(\psi(\tilde{x}) - \psi(x)) + \psi'(x)(x^2 - \tilde{x}^2))}{(\psi(\tilde{x}) - \psi(x))^2},$$

$$\zeta_{\tilde{x}} = \frac{\psi(x)(\psi'(\tilde{x})(\tilde{x}^2 - x^2) - 2\tilde{x}(\psi(\tilde{x}) - \psi(x)))}{(\psi(\tilde{x}) - \psi(x))^2},$$

$$\frac{d\tilde{x}}{dx} = \frac{\Phi'(x)}{\Phi'(\tilde{x})} = \frac{\psi(x)}{\psi(\tilde{x})} < 0.$$

Recalling $\psi(x) < 0, \psi(\tilde{x}) > 0, \psi'(x) > 0, \tilde{x}^2 - x^2 = (\tilde{x} - x)(\tilde{x} + x) > 0$, hence we have $\zeta_x < 0$. Now we will prove that $\zeta_{\tilde{x}} > 0$ to obtain $\zeta'(x) < 0$ and therefore $Q'(h) > 0$. It is sufficient to prove that

$$E(x) := \psi'(\tilde{x})(\tilde{x}^2 - x^2) - 2\tilde{x}(\psi(\tilde{x}) - \psi(x)) < 0. \tag{15}$$

To this end, we compute the resultant w.r.t $\tilde{x}$ of the above expression and $(\Phi(\tilde{x}) - \Phi(x))/(\tilde{x} - x)$ with respect to $\tilde{x}$, and obtain

$$Res\left[\psi'(\tilde{x})(\tilde{x}^2 - x^2) - 2\tilde{x}(\psi(\tilde{x}) - \psi(x)), \frac{\Phi(\tilde{x}) - \Phi(x)}{\tilde{x} - x}, \tilde{x}\right] = -10584x^{30}(x-1)^2(6x-7)p_9(x),$$

where

$$p_9(x) = -135485 - 2450x - 108472x^2 + 319640x^3 - 409104x^4 - 95328x^5 + 1534464x^6 - 767232x^7 - 946944x^8 + 594432x^9.$$

By applying Sturm's Theorem, we find that $p_9(x) \neq 0$ for all $x \in (\alpha, 0)$, with $\alpha \approx -0.67$ where $H(\alpha, 0) = H(1, 0) = \frac{1}{42}$. This implies that the function $E(x)$ defined in (15) is strictly negative for all $x \in [\alpha, 0)$, since $E(\alpha) \approx -1.00204 < 0$.

(ii) We will prove now the second statement, namely $P'(h) > 0$. By the definition of $\mu(h)$ and $\nu(h)$, we have $H(\mu(h), 0) \equiv H(\nu(h), 0) = h$ and hence $\Phi(\mu(h)) \equiv \Phi(\nu(h)) = h$. Thus, $7(\nu(h)^6 - \mu(h)^6) = 6(\nu(h)^7 - \mu(h)^7)$. This yields

$$\begin{aligned} 6(\mu(h)^6 &+ \mu(h)^5\nu(h) + \mu(h)^4\nu(h)^2 + \mu(h)^3\nu(h)^3 \\ &+ \mu(h)^2\nu(h)^4 + \mu(h)\nu(h)^5 + \nu(h)^6) \\ &= 7(\nu(h) + \mu(h))(\nu(h)^2 + \mu(h)^2 \\ &- \mu(h)\nu(h))(\nu(h)^2 + \mu(h)^2 + \mu(h)\nu(h)). \end{aligned} \tag{16}$$

Let us define $U(h) = \mu(h) + \nu(h)$ and $\eta(h) = \mu(h)\nu(h) < 0$. Then

$$\begin{aligned} 6(U(h)^6 &- 5\eta(h)U(h)^4 + 6\eta(h)^2U(h)^2 - \eta(h)^3) \\ &= 7U(h)(U(h)^2 - 3\eta(h))(U(h)^2 - \eta(h)). \end{aligned}$$

Now we consider, for a fixed $U_0 \in (0, 0.33)$, the polynomial $F(\eta; U_0)$ of degree three in η as

$$\begin{aligned} F(\eta; U_0) := &\ 6(U_0^6 - 5U_0^4\eta + 6U_0^2\eta^2 - \eta^3) \\ &- 7U_0(U_0^2 - 3\eta)(U_0^2 - \eta). \end{aligned}$$

It is easy to see that

$$F(0; U_0) = U_0^5(-7 + 6U_0) < 0,$$
$$F(U_0^2; U_0) = 6U_0^6 > 0, \ F(+\infty; U_0) = -\infty < 0.$$

This means that equation $F(\eta; U_0) = 0$ for η has at least two positive real roots, and hence at most one negative real root. Suppose that $\eta^* < 0$ be the only negative root of equation $F(\eta; U_0) = 0$. Then it corresponds to some $h^* \in (0, 1/42)$ such that $\eta(h^*) = \eta^*$ and $U(h^*) = U_0$. The geometric interpretation of this fact is that each straight line $\{U = U_0\}$ in the (h, U)-plane intersects the graph of $U = U(h)$ at most at one point such as h^* (counting multiplicity). Therefore, due to the fact that $U'(0^+) = U'(\frac{1}{42}^-) = +\infty$, we have $U'(h) = \mu'(h) + \nu'(h) > 0$ for all $h \in (0, \frac{1}{42})$. Now the result follows from Theorem 2.1 of [3]. □

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