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On the distribution of limit cycles in a Li'enard system with a nilpotent center and a nilpotent saddle

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On the distribution of limit cycles in a Liénard system with a nilpotent center and a nilpotent saddle

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Abstract

In this work, we study the Abelian integral $I(h)$ corresponding to the following Liénard system,

$$\dot{x} = y, \ \ \dot{y} = x^3(x-1)^3 + \varepsilon(a + bx + cx^3 + x^5)y,$$

where $0 < \varepsilon \ll 1$, a, b and c are real bounded parameters. By using the expansion of $I(h)$ and a new algebraic criterion developed in [Grau et al., 2011], it will be shown that the sharp upper bound of the maximal number of isolated zeros of $I(h)$ is 4. Hence, the above system can have at most 4 limit cycles bifurcating from the corresponding period annulus. Moreover, the configuration (distribution) of the limit cycles is also determined. The results obtained are new for this kind of Liénard system. MSC: 34C07; 34C08; 37G15; 34M50

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1 Introduction and statement of the results

The second part of the famous Hilbert 16th problem is to find an upper bound to the maximal number and to determine configurations of limit cycles of planar polynomial differential systems defined by

$$\frac{dx}{dt} = P_n(x, y), \qquad \frac{dy}{dt} = Q_n(x, y),$$

for all possible P_n and Q_n where P_n and Q_n are real polynomials in x, y of degree n. The maximal number of limit cycles is usually denoted by $H(n)$, and it is so called the Hilbert number.

In general, determination of $H(n)$ is a very difficult problem. Keeping this in mind, Arnold proposed in [Arnold, 1990] ten problems, that it is named the 7th problem as follows:

Consider a deformation of a planar Hamiltonian system as follows:

$$dH + \varepsilon(f dx + g dy) = 0, \tag{1}$$

where f and g are polynomials in x, y of degrees at most n, $\varepsilon > 0$ is a small parameter. Abelian integrals related to deformation (1) are defined by

$$I(h) = \oint_{L_h} f(x, y)dx + g(x, y)dy,$$

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where

$$L_h := \{(x,y) \in \mathbb{R}^2 \ : \ H(x,y) = h, \ h_1 < h < h_2\}.$$

The Arnold 7th problem (the first part): How many isolated zeros the Abelian integrals $I(h)$ would have in the domain of definition if f, g and H are polynomials with known degrees?

The first part of the above-mentioned problem is called the weakened (infinitesimal or tangential) Hilbert 16th problem.

This article studies the number and distribution of limit cycles of a polynomial Liénard system with a nilpotent center and a nilpotent saddle connection. The problem considered here is related to the still open Hilbert 16th problem.

Now let us clarify the results of this article. Consider a perturbed planar Hamiltonian system of the form

$$\begin{aligned} \dot{x} &= H_y + \varepsilon p(x,y,\varepsilon,\delta) \\ \dot{y} &= -H_x + \varepsilon q(x,y,\varepsilon,\delta) \end{aligned} \tag{2}$$

where p, q and H are C^ω functions, ε is a small positive parameter and δ is a vector parameter where $\delta \in D \subset \mathbb{R}^m$ and D is a compact set. Note that $(2)|_{(\varepsilon=0)}$ is a Hamiltonian system.

Now suppose $(2)|_{(\varepsilon=0)}$ has a family of periodic orbits

$$L_h := \{(x,y) \in \mathbb{R}^2 \ : \ H(x,y) = h, \ h_1 < h < h_2\}$$

continuously depending on the parameter h. Associated to periodic orbits L_h, we can define the function

$$M(h,\delta) := \oint_{L_h} q dx - p dy|_{\varepsilon=0},$$

which is called the first order Melnikov function or Abelian integral [Ilyashenko, 2002], and it is very important in the study of bifurcations of limit cycles in the family (2).

In this work, the following Newtonian system will be considered

$$\dot{x} = y, \qquad \dot{y} = x^3(x-1)^3, \tag{H_0}$$

with the Hamiltonian function

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{7}x^7 + \frac{1}{2}x^6 - \frac{3}{5}x^5 + \frac{1}{4}x^4, \tag{3}$$

which has a nilpotent saddle at $S(1,0)$, a nilpotent center at $O(0,0)$ and a saddle connection $L_{\frac{1}{140}}$.

All orbits L_h inside $L_{\frac{1}{140}}$, for $h \in \left(0, \frac{1}{140}\right)$, are closed.

We will study a perturbation of (H_0) in the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x^3(x-1)^3 + \varepsilon(a + bx + cx^3 + x^5)y, \end{aligned} \tag{H_ε}$$

and especially, the related Abelian integral:

$$I(h,\delta) = \oint_{L_h} (a + bx + cx^3 + x^5) y dx = aI_0(h) + bI_1(h) + cI_3(h) + I_5(h), \tag{4}$$

where $I_i(h) = \oint_{L_h} x^i y dx$ $(i = 0,1,3,5)$ and L_h is oriented clockwise. Here $0 < \varepsilon \ll 1$ and $\delta = (a,b,c)$ belongs to any compact subset of $\mathbb{R}^3$.

Our main result is the following.

Theorem 1.1. *The exact upper bound for the maximal number of isolated zeros of $I(h,\delta)$, defined in* (4)*, is 4 on the open interval $(0, \frac{1}{140})$. Hence, the Liénard system (H_ε) can have at most 4 limit cycles bifurcating from the corresponding period annulus. Moreover, the distribution of the limit cycles of system (H_ε) is illustrated in Figure 2.*

The main theorem will be deduced by proving four propositions in Section 3 and one lemma in Section 4. For more details, see the Propositions 3.2-3.5 in Section 3, Lemma 4.3 in Section 4, and finally Corollary 4.4 at the end of the present paper. Before that, we give some preliminaries which are used to find the number of limit cycles bifurcating from the period annulus $\{L_h\}_h$ in Section 2.

2 Preliminary

In this section, some definitions will be provided and some lemmas will be recalled which will be used in the proof of the main results.

Definition 2.1. Let $f_0, f_1, ..., f_{n-1}$ be real analytic functions on some open interval $\mathbb{I}$ of $\mathbb{R}$. Then

(i) The set $\{f_0, f_1, ..., f_{n-1}\}$ is called a Chebyshev space on $\mathbb{I}$ if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + ... + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on $\mathbb{I}$.

(ii) The set $\{f_0, f_1, ..., f_{k-1}\}$ is called a complete Chebyshev space on $\mathbb{I}$ if for all $k = 1, 2, ..., n$, any nontrivial linear combination of $(f_0, f_1, ..., f_{k-1})$ as

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + ... + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on $\mathbb{I}$.

(iii) The set $\{f_0, f_1, ..., f_{k-1}\}$ is called an extended complete Chebyshev space on $\mathbb{I}$ if for all $k = 1, 2, ..., n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + ... + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on $\mathbb{I}$ counting multiplicity.

Remark 2.2. Recall that a real vector space V of real analytic functions defined on some real interval $\mathbb{I}$ is considered to be Chebyshev, provided that each f in V has at most $dimV - 1$ isolated zeros (counted with multiplicity) on $\mathbb{I}$, and Chebyshev with accuracy m, when each f in V has at most $dimV + m - 1$ isolated zeros at hand.

Definition 2.3. Let $f_0, f_1, ..., f_{k-1}$ be real analytic functions on some open interval $\mathbb{I}$ of $\mathbb{R}$. The Wronskian of $(f_0, f_1, ..., f_{k-1})$ at $x \in \mathbb{I}$ is

$$W[f_0, f_1, ..., f_{k-1}](x) = \begin{vmatrix} f_0 & f_1 & \cdots & f_{k-1} \\ f_0' & f_1' & \cdots & f_{k-1}' \\ \vdots & \vdots & \vdots & \vdots \\ f_0^{(k-1)} & f_1^{(k-1)} & \cdots & f_{k-1}^{(k-1)} \end{vmatrix}.$$

The following result concerning an extended complete Chebyshev space is well known.

Lemma 2.4. *The set $\{f_0, f_1, ..., f_{n-1}\}$ is an extended complete Chebyshev space on $\mathbb{I}$ if and only if, for each $k = 1, 2, ..., n$,*

$$W[f_0, f_1, ..., f_{k-1}](x) \neq 0 \quad for\ all \quad x \in \mathbb{I}.$$

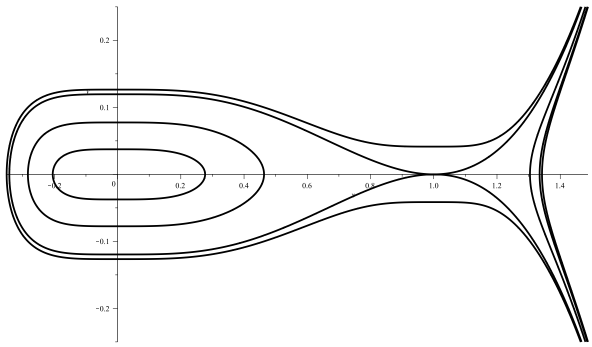

Figure 1: The level sets of (H_0).

Suppose $(2)|_{\varepsilon=0}$ has a nilpotent center $O(0,0)$, a homoclinic loop L_{h_s} passing through a nilpotent saddle $S(x_s,y_s)$ and surrounding $O(0,0)$. We suppose that the periodic orbits are oriented clockwise and closed curves inside the homoclinc loop L_{h_s} are parameterized by the Hamiltonian value h such that $H(x_s,y_s)=h_s$ corresponds to a homoclinc loop L_{h_s} and $H(0,0)=0$ corresponds to a nilpotent center (Figure 1).

Correspondingly, we have

$$I(h,\delta) \quad = \quad \oint_{L_h} qdx - pdy, \qquad h\in(0,h_s).$$

By using the notations of [Jiang & Han, 2008; Han et al., 2008], when $O(0,0)$ is a nilpotent center we can suppose

$$H(x,y)=\frac{y^2}{2}+\bar{h}_{40}x^4+\sum_{i+j\geq 4}\bar{h}_{ij}x^iy^j, \qquad \bar{h}_{40}>0, \tag{5}$$

and

$$p(x,y,\delta)=\sum_{i+j>0}\bar{a}_{ij}(\delta)x^iy^j, \quad q(x,y,\delta)=\sum_{i+j>0}\bar{b}_{ij}(\delta)x^iy^j.$$

For the expansion of $I(h,\delta)$ near the nilpotent center $O(0,0)$ we have from [Jiang & Han, 2008] that

$$I(h,\delta)=h^{\frac{3}{4}}\sum_{k\geqslant 0}b_k(\delta)h^{\frac{k}{2}}, \quad \text{as} \quad h\longrightarrow 0^+. \tag{6}$$

Under (5), the coefficients b_j, $j=0,1,2,...,n$, can be obtained by following the method in [Jiang & Han, 2008].

Now let the outer boundary of the family $\{L_h\}_{h\in(0,h_s)}$ be a homoclinic loop given by

$$L_s := L_{h_s} = \{(x,y)\in\mathbb{R}^2 : H(x,y)=h_s\},$$

that is homoclinic (connected) to a nilpotent saddle at (x_s,y_s). Then by [Zang et al., 2008], it is possible to write

$$H(x,y)=h_s-\frac{1}{4}(x-x_s)^4+\sum_{i\geq 5}h_{i0}(x-x_s)^i+(y-y_s)^2\sum_{i+j\geq 0}h_{ij}(x-x_s)^i(y-y_s)^j. \tag{7}$$

Also, we can write

$$p(x,y,\delta)=\sum_{i+j\geq 0} a_{ij}(\delta)(x-x_s)^i(y-y_s)^j,\quad q(x,y,\delta)=\sum_{i+j\geq 0} b_{ij}(\delta)(x-x_s)^i(y-y_s)^j.$$

The following asymptotic expansion of the Abelian integral $I(h,\delta)$ is obtained in [Zang et al., 2008] near the nilpotent saddle loop.

Lemma 2.5. *([Zang et al., 2008]) Let (7) hold. Then for system* (2), *near the critical value $h=h_s$ ($0<h_s-h\ll 1$) corresponding to the nilpotent saddle loop L_s through the point (x_s,y_s) as a nilpotent saddle, we have*

$$\begin{aligned} I(h,\delta) &= c_0+c_1|h-h_s|^{\frac{3}{4}}+c_2(h-h_s)\ln|h-h_s|+c_3|h-h_s|+c_4|h-h_s|^{\frac{5}{4}}+c_5|h-h_s|^{\frac{7}{4}}\\ &+ c_6(h-h_s)^2\ln|h-h_s|+O((h-h_s)^2),\quad as\ \ h\longrightarrow h_s^-, \end{aligned} \tag{8}$$

with

$$\begin{aligned} &c_0=I(h_s,\delta)=\oint_{L_s} qdx-pdy,\qquad c_1=-\frac{4\sqrt{2}d_{0,0}\Delta_{0,2}}{3},\qquad c_2=\frac{\sqrt{2}d_{1,0}}{2},\\ &c_3=\oint_{L_s}(p_x+q_y-a_{10}-b_{01})dt\qquad for\ \ c_1=c_2=0,\qquad c_4=-\frac{4\sqrt{2}d_{2,0}\Delta_{2,2}}{5},\\ &c_5=\frac{8\sqrt{2}(d_{0,2}-2d_{4,0})\Delta_{0,2}}{21},\qquad c_6=\frac{\sqrt{2}(4d_{5,0}-d_{1,2})}{8}, \end{aligned}$$

where $\Delta_{0,2}>0$ and $\Delta_{2,2}<0$ are real constants and $d_{i,0}$ ($i=0,1,2,4,5$), $d_{i,2}$ ($i=0,1$), are some coefficients depending on h_{ij}, $a_{i,j}$ and $b_{i,j}$ ($0\leq i,j\leq 7$). The coefficients c_2,c_3,c_5,c_6 are called local coefficients of system (2) *at the nilpotent saddle.*

The next theorem follows by modifying the proof of part (ii) of Theorem 1.5 in [Sun et al., 2011].

Theorem 2.6. *Consider the expansions* (6) *and* (8) *of $I(h,\delta)$ for system* (2). *If there exists some $\delta_0\in\mathbb{R}^m$ such that*

$$\begin{aligned} c_0(\delta_0)=c_1(\delta_0)=...=c_{k_1-1}(\delta_0)=0,\qquad c_{k_1}(\delta_0)\neq 0,\\ b_0(\delta_0)=b_1(\delta_0)=...=b_{k_2-1}(\delta_0)=0,\qquad b_{k_2}(\delta_0)\neq 0, \end{aligned}$$

and

$$rank\frac{\partial(c_0,c_1,...,c_{k_1-1,b_0,b_1,...,b_{k_2-1}})}{\partial(\delta_1,...,\delta_m)}(\delta_0)=k_1+k_2,$$

then we would have $k_1+k_2+\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2}$ limit cycles for some (ε,δ) near $(0,\delta_0)$, from which k_1 limit cycles are near the homoclinic loop $L_{\frac{1}{140}}$, k_2 limit cycles are near the center $O(0,0)$ and $\frac{1-sgn(M(h_1,\delta_0)M(h_2,\delta_0))}{2}$ limit cycle are surrounding the center $O(0,0)$, where $h_1=h_s-\varepsilon_1$, $h_2=0+\varepsilon_2$ with ε_1 and ε_2 are positive and very small.

3 Asymptotic expansions of the Melnikov function I(h,δ)

In this section we will apply Lemma 2.5 to system (H_ε) and will obtain the asymptotic expansion of the Abelian integral $I(h,\delta)$ in (4) as $h\to(\frac{1}{140})^-$. It is clear that on the loop $L_{\frac{1}{140}}$ we have that

$$y^{\pm}(x)=\pm\frac{1}{70}\sqrt{1400\,x^3+700\,x^2+280\,x+70}\,(x-1)^2\,.$$

By applying the formula for $c_0(\delta)$ in Lemma 2.5, we have

$$\begin{aligned} c_0 &= \oint_{L_{\frac{1}{140}}} q dx - p dy = 2\int_{x_l=-0.3423840948}^{1} (a+bx+cx^3+x^5)y^+(x)dx \\ &= -18.28103849\, a - 6.470189824\, b - 0.2839284832\, c + 0.01549000622. \end{aligned}$$

To compute the expressions for the local coefficients c_1, c_2 described in Lemma 2.5, we must translate the saddle point $S = (1, 0)$ to the origin. To this end, we make the transformations $X = 1-x$, $Y = y$ and $T = -t$ and still denote X, Y and T by x, y and t, respectively. Then system (H_ε) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x^3(x-1)^3 + \varepsilon q(x,y), \end{aligned} \tag{9}$$

where

$$q(x,y) = (a+c+b+1-(b+3\,c+5)\,x + (10+3\,c)\,x^2 - (c+10)\,x^3 + 5\,x^4 - x^5)y.$$

The Hamiltonian function of system $(9)|_{\varepsilon=0}$ is defined as

$$\tilde{H}(x,y) = \frac{1}{2}\,y^2 - \frac{1}{4}\,x^4 + \frac{3}{5}\,x^5 - \frac{1}{2}\,x^6 + \frac{1}{7}\,x^7.$$

Thus, by using Lemma 2.5, it is founded that

$$\begin{aligned} \Delta_{0,2} &= 1.311028778\sqrt{2}, \\ d_{0,0} &= a+b+c+1, \\ d_{1,0} &= \frac{6}{5}a + \frac{1}{5}b - \frac{9}{5}c - \frac{19}{5}, \\ c_1 &= -3.496076742\, a - 3.496076742\, b - 3.496076742 - 3.496076742\, c, \\ c_2 &= \frac{1}{10}\sqrt{2}\,(6\,a + b - 19 - 9\,c)\,. \end{aligned}$$

If $c_1 = c_2 = 0$ and hence $a = 4 + 2\,c$ and $b = -3\,c - 5$, then we would have

$$\begin{aligned} c_3 &= \oint_{L_{\frac{1}{140}}} (p_x + q_y)dt = 140\int_{-0.3423840948}^{1} \frac{(a+bx+cx^3+x^5)}{(x-1)^2\sqrt{1400\,x^3+700\,x^2+280\,x+70}}dx \\ &= -1773.900826\, c - 4170.105830. \end{aligned}$$

For the expansion of the Melnikov function (4) of system (H_0) near the nilpotent center $L_0 = \{(0,0)\}$, we will prove the next result.

Lemma 3.1. *The Abelian integral $I(h,\delta)$ given in (4) has the asymptotic expansion*

$$I(h,\delta) = h^{\frac{3}{4}} \sum_{k\geqslant 0} b_k(\delta) h^{\frac{k}{2}}, \tag{10}$$

near the nilpotent center $(0,0)$, where $0 < h \ll 1$ and $\delta = (a,b,c)$. Following the method in [Jiang & Han, 2008], the first coefficients of the above expansion are given by

$$\begin{aligned} b_0 &= 2B\left(\frac{1}{4}, \frac{3}{2}\right) a, \\ b_1 &= \frac{12}{25}B\left(\frac{3}{4}, \frac{3}{2}\right)(15\,b + 19\,a)\,, \\ b_2 &= \frac{2}{875}B\left(\frac{5}{4}, \frac{3}{2}\right)(10500\,c + 28960\,b + 36873\,a)\,, \\ b_3 &= \frac{4}{3125}B\left(\frac{7}{4}, \frac{3}{2}\right)(52500 + 243500\,c + 190343\,b)\,. \end{aligned}$$

Proof. Considering the oval L_h for $0 < h \ll 1$, we denote the abscissas of the intersection points of L_h with the negative and positive half x-axis by $x_l(h)$ and $x_r(h)$, respectively. Thus, $x_l(h) < 0 < x_r(h) < 1$ and $A(x_l(h)) = A(x_r(h)) \equiv h$, where

$$A(x) = x^4 \left(-\frac{1}{7}\, x^3 + \frac{1}{2}\, x^2 - \frac{3}{5}\, x + \frac{1}{4} \right).$$

It can be calculated that

$$x_r(h) = g^{-1}(\sqrt[4]{h}) = \sqrt{2}\sqrt[4]{h} + \frac{6}{5}\sqrt{h} + \frac{38}{25}\sqrt{2}h^{3/4} + \frac{4196}{875}h + O(h^{5/4}) > 0,$$
$$x_l(h) = g^{-1}(-\sqrt[4]{h}) = -\sqrt{2}\sqrt[4]{h} + \frac{6}{5}\sqrt{h} - \frac{38}{25}\sqrt{2}h^{3/4} + \frac{4196}{875}h + O(h^{5/4}) < 0,$$

where $g(x) = sign(x)\sqrt[4]{A(x)}$. Define a new function as

$$F(x,z) := x\left(-\frac{1}{7}\, x^3 + \frac{1}{2}\, x^2 - \frac{3}{5}\, x + \frac{1}{4} \right)^{\frac{1}{4}} - z, \qquad (x,z) \in (\mathbb{R}^2, 0).$$

The implicit function theorem implies that there exists a smooth function $x = \psi(z)$ for $|z| \ll 1$ such that $F(\psi(z), z) \equiv 0$. By using Maple 18, it can be deduced that for $0 < |z| \ll 1$,

$$x = \psi(z) = \sqrt{2}z + 6/5\, z^2 + \frac{38}{25}\sqrt{2}z^3 + \frac{4196}{875}\, z^4 + \frac{36873}{4375}\sqrt{2}z^5 + O\left(z^6\right).$$

Under the transformation

$$A(x) = z^4, \qquad or \;\; x\left(-\frac{1}{7}\, x^3 + \frac{1}{2}\, x^2 - \frac{3}{5}\, x + \frac{1}{4} \right)^{\frac{1}{4}} = z, \tag{11}$$

the Abelian integral (4) changes into

$$\begin{aligned} I(h,\delta) &= 2\sqrt{2}\int_{x_l(h)}^{x_r(h)} (a + bx + cx^3 + x^5)\sqrt{h - A(x)}dx \\ &= 2\sqrt{2}\int_{-h^{\frac{1}{4}}}^{h^{\frac{1}{4}}} (a + bx + cx^3 + x^5)\mid_{x=\psi(z)} \sqrt{h - z^4}\psi'(z)dz \\ &= 2\sqrt{2}\sum_{k=0}^{+\infty} a_k(\delta)E_k(h), \end{aligned} \tag{12}$$

where

$$E_k(h) = \int_{-h^{\frac{1}{4}}}^{h^{\frac{1}{4}}} z^k\sqrt{h - z^4}dz, \quad \text{for} \;\; k = 0, 1, 2, \cdots,$$

and

$$\begin{aligned} a_0 &= a\sqrt{2}, \\ a_1 &= \frac{2}{5}\,(6\,a + 5\,b)\,, \\ a_2 &= \frac{6}{25}\sqrt{2}\,(15\,b + 19\,a)\,, \\ a_3 &= \frac{4}{875}\,(875\,c + 4196\,a + 3290\,b)\,, \\ a_4 &= \frac{1}{875}\sqrt{2}\,(10500\,c + 28960\,b + 36873\,a)\,, \\ a_5 &= \frac{4}{875}\,(32709\,b + 13650\,c + 1750)\,, \\ a_6 &= \frac{2}{3125}\sqrt{2}\,(52500 + 243500\,c + 190343\,b)\,. \end{aligned}$$

Therefore in order to get the asymptotic expansion of $I(h,\delta)$ near $h=0$, it is crucial to compute the following elliptic integrals

$$E_k(h)=\int_{-h^{\frac{1}{4}}}^{h^{\frac{1}{4}}} z^k\sqrt{h-z^4}dz=\frac{1+(-1)^k}{4}B\left(\frac{k+1}{4},\frac{3}{2}\right)h^{\frac{3+k}{4}},$$

where $k=0,1,2,...$ and $B(\alpha,\beta)$ is the Beta-function for $\alpha>0$, $\beta>0$, that is

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt=\frac{1}{\alpha}+\sum_{n=1}^{+\infty}\frac{(-1)^n(\beta-1)(\beta-2)...(\beta-n)}{n!(\alpha+n)}.$$

Let $z=h^{\frac{1}{4}}s$, where $s\in[-1,1]$. Then

$$E_k(h)=h^{\frac{3+k}{4}}\int_{-1}^1 s^k\sqrt{1-s^4}ds=h^{\frac{3+k}{4}}\int_0^1[1+(-1)^k]s^k\sqrt{1-s^4}ds.$$

Using the mapping $s^4=w$, we further obtain

$$E_k(h)=\frac{1+(-1)^k}{4}h^{\frac{3+k}{4}}\int_0^1 w^{\frac{k-3}{4}}(1-w)^{\frac{1}{2}}dw=\frac{1+(-1)^k}{4}B\left(\frac{k+1}{4},\frac{3}{2}\right)h^{\frac{3+k}{4}}.$$

It is obvious that $E_k(h)\equiv 0$ when k is odd. We get the elliptic integrals for $k=0,2,4,6$ as follows :

$$E_0=\frac{1}{2}B\left(\frac{1}{4},\frac{3}{2}\right)h^{\frac{3}{4}},\quad E_2=\frac{1}{2}B\left(\frac{3}{4},\frac{3}{2}\right)h^{\frac{5}{4}},\quad E_4=\frac{1}{2}B\left(\frac{5}{4},\frac{3}{2}\right)h^{\frac{7}{4}},\quad E_6=\frac{1}{2}B\left(\frac{7}{4},\frac{3}{2}\right)h^{\frac{9}{4}}.$$

Substituting the above terms into (12), the result follows. □

Proposition 3.2. *There exists some values (a,b,c,ε) for which the system (H_ε) can have 3 limit cycles near the center $L_0=\{(0,0)\}$.*

Proof. It is easily seen that the set of equations $\{b_0(\delta)=b_1(\delta)=b_2(\delta)=0\}$ has a unique solution at $\delta_0=(a,b,c)=(0,0,0)$. By substituting this into $b_3(\delta)$ and $c_0(\delta)$ we find that $b_3(\delta_0)=21.47067301>0$, $c_0(\delta_0)=0.01549000622>0$. Hence, for $h_1=0+\varepsilon_1$, $h_2=\frac{1}{140}-\varepsilon_2$ with ε_1 and ε_2 positive and very small, we have that

$$\begin{aligned}&I(h_1,\delta_0)=b_3(\delta_0)h_1^{\frac{9}{4}}+O(h_1^{\frac{11}{4}})>0,\\&I(h_2,\delta_0)=c_0(\delta_0)+o(1)>0,\quad \text{as}\quad h_2\longrightarrow h_s,\\&\frac{1-sgn(I(h_1,\delta_0)I(h_2,\delta_0))}{2}=0.\end{aligned}$$

In addition, one can easily verify that

$$\text{rank}\frac{\partial(b_0(\delta),b_1(\delta),b_2(\delta))}{\partial(a,b,c)}(\delta_0)=3.$$

Therefore, according to Theorem 2.6, there exists some (a,b,c,ε) near $(0,0,0,0)$ such that system (H_ε) has 3 limit cycles near the center L_0. □

Proposition 3.3. *There exists some values (a,b,c,ε) for which the system (H_ε) can have 3 limit cycles near the saddle loop $L_{\frac{1}{140}}$.*

Proof. It is easily seen that the set of equations $\{c_0(\delta) = c_1(\delta) = c_2(\delta) = 0\}$ has a unique solution at $\delta_0 = (a, b, c) = (-0.6752733748, 2.012910062, -2.337636687)$. By substituting this into $b_0(\delta)$ and $c_3(\delta)$ we find that $b_0(\delta_0) = -4.721615076 > 0$, $c_3(\delta_0) = -23.370180 < 0$. Hence, for $h_1 = 0 + \varepsilon_1$, $h_2 = \frac{1}{140} - \varepsilon_2$ with ε_1 and ε_2 positive and very small, we have that

$$\begin{aligned}
&I(h_1, \delta_0) = b_0(\delta_0) h_1^{\frac{3}{4}} + O(h_1^{\frac{5}{4}}) < 0, \\
&I(h_2, \delta_0) = c_3(\delta_0)|h_2 - h_s| + O(|h_2 - h_s|^{\frac{5}{4}}) < 0, \\
&\frac{1 - sgn(I(h_1, \delta_0) I(h_2, \delta_0))}{2} = 0.
\end{aligned}$$

Moreover, one can easily check that

$$\operatorname{rank} \frac{\partial(c_0(\delta), c_1(\delta), c_2(\delta))}{\partial(a, b, c)}(\delta_0) = 3.$$

Therefore, on account of Theorem 2.6, there exists some (a, b, c, ε) near

$$(-0.6752733748, 2.012910062, -2.337636687, 0)$$

such that system (H_ε) has 3 limit cycles near the saddle loop $L_{\frac{1}{140}}$. $\square$

Proposition 3.4. *There exists some values (a, b, c, ε) for which the system (H_ε) can have 4 limit cycles, from which one limit cycle is near the nilpotent center L_0, two limit cycles are near the saddle loop $L_{\frac{1}{140}}$ and the fourth limit cycle lies between the center L_0 and the homoclinic loop $L_{\frac{1}{140}}$.*

Proof. It is easily seen that the set of equations $\{c_0(\delta) = c_1(\delta) = b_0(\delta) = 0\}$ has a unique solution at $\delta_0 = (a, b, c) = (0, 0.04840055616, -1.048400556)$. By substituting this into $b_1(\delta)$ and $c_2(\delta)$ we find that $b_1(\delta_0) = 0.3340261655 > 0$, $c_2(\delta_0) = -.9515994436\sqrt{2} < 0$. Hence, for $h_1 = 0 + \varepsilon_1$, $h_2 = \frac{1}{140} - \varepsilon_2$ with ε_1 and ε_2 positive and very small, we have

$$\begin{aligned}
&I(h_1, \delta_0) = b_1(\delta_0) h_1^{\frac{5}{4}} + O(h_1^{\frac{7}{4}}) > 0, \\
&I(h_2, \delta_0) = c_2(\delta_0)(h_2 - h_s) \ln |h_2 - h_s| + O(|h_2 - h_s|) < 0, \\
&\frac{1 - sgn(I(h_1, \delta_0) I(h_2, \delta_0))}{2} = 1.
\end{aligned}$$

On top of that, it is easily seen that

$$\operatorname{rank} \frac{\partial(c_0(\delta), c_1(\delta), b_0(\delta))}{\partial(a, b, c)}(\delta_0) = 3.$$

Therefore, due to Theorem 2.6, there exists some (a, b, c, ε) near $(0, 0.04840055616, -1.048400556, 0)$ such that system (H_ε) has 4 limit cycles, from which one limit cycle is near the nilpotent center L_0, two limit cycles are near the saddle loop $L_{\frac{1}{140}}$ and the fourth limit cycle lies between the center L_0 and the homoclinic loop $L_{\frac{1}{140}}$. $\square$

Proposition 3.5. *There exists some values (a, b, c, ε) for which the system (H_ε) can have 4 limit cycles, from which one limit cycle is near the homoclinic loop $L_{\frac{1}{140}}$, two limit cycles are near the nilpotent center L_0 and the last limit cycle lies between the center L_0 and the homoclinic loop $L_{\frac{1}{140}}$.*

Proof. It is easily seen that the set of equations $\{b_0(\delta) = b_1(\delta) = c_0(\delta) = 0\}$ has a unique solution at $\delta_0 = (a, b, c) = (0, 0, 0.05455601370)$. By substituting this into $b_2(\delta)$ and $c_1(\delta)$ we find that

$b_2(\delta_0) = 0.6539383216 > 0$, $c_1(\delta_0) = -3.686808753 < 0$. Hence, for $h_1 = 0 + \varepsilon_1$, $h_2 = \frac{1}{140} - \varepsilon_2$ with ε_1 and ε_2 positive and very small, we get

$$
\begin{aligned}
&I(h_1, \delta_0) = b_2(\delta_0) h_1^{\frac{7}{4}} + O(h_1^{\frac{9}{4}}) > 0,\\
&I(h_2, \delta_0) = c_1(\delta_0)|h_2 - h_s|^{\frac{3}{4}} + O(|h_2 - h_s| \ln |h_2 - h_s|) < 0,\\
&\frac{1 - sgn(I(h_1, \delta_0) I(h_2, \delta_0))}{2} = 1.
\end{aligned}
$$

Moreover, one can easily prove that

$$\text{rank} \frac{\partial(b_0(\delta), b_1(\delta), c_0(\delta))}{\partial(a, b, c)}(\delta_0) = 3.$$

Therefore, by using Theorem 2.6, there exists some (a, b, c, ε) near $(0, 0, 0.05455601370, 0)$ such that system (H_ε) has 4 limit cycles, from which one limit cycle is near the homoclinic loop $L_{\frac{1}{140}}$, two limit cycles are near the nilpotent center L_0 and the last limit cycle lies between the center L_0 and the homoclinic loop $L_{\frac{1}{140}}$. $\square$

For the distribution of the limit cycles obtained above, see Figure 2.

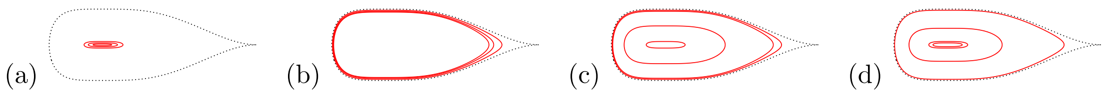

Figure 2: The distribution of the limit cycles bifurcated from the period annulus of system (H_0).

4 The number of zeros of the Abelian integral I(h)

In this section, we will study the maximum number of limit cycles that can bifurcate from the period annulus of system (H_0) for $0 < \varepsilon \ll 1$. We will use an algebraic criterion developed in [Grau et al., 2011] to study the Abelian integral $I(h)$ of system (H_ε). In fact, it will be shown that the base functions $\{I_0(h),\ I_1(h),\ I_3(h),\ I_5(h)\}$ in the Abelian integral $I(h)$ form a Chebyshev system with accuracy 1. Hence, the number of isolated zeros of $I(h)$ in the open interval $(0, \frac{1}{140})$ is at most four.

We notice that the Abelian integrals $I_i(h)$ of (4) take the form $I_i(h) = \oint_{\gamma_h} x^i y^{2s-1} dx,\ i = 0, 1, 3, 5$ with $s = 1$ and the Hamiltonian function $H(x, y)$ in (3) takes the form $H(x, y) = A(x) + B(x)y^{2m}$ where $A(x) = -\frac{1}{7}x^7 + \frac{1}{2}x^6 - \frac{3}{5}x^5 + \frac{1}{4}x^4$, $B(x) = \frac{1}{2}$ and $m = 1$. The projection of the period annulus on the x-axis is $(x_l, 1)$ and $xA'(x) > 0$ for all $x \in (x_l, 1) \setminus \{0\}$, here

$$x_l = -\frac{1}{30} \frac{\left(350 + 105\sqrt{15}\right)^{2/3} - 35 + 5\sqrt[3]{350 + 105\sqrt{15}}}{\sqrt[3]{350 + 105\sqrt{15}}} \in \left(-\frac{351}{1024}, -\frac{175}{512}\right). \tag{13}$$

Therefore, there exists an involution function $z(x)$ with $x_l < z(x) < 0$ such that $A(x) = A(z(x))$ for all $0 < x < 1$. Recall that a C^1 mapping $z : I \longrightarrow I$ is an involution when $z^2 = i.d.$ and $z \neq i.d.$ Note that an involution is a diffeomorphism with a unique fixed point.
Now we restate Theorem A of [Mañosas & Villadelprat, 2011] and Lemma 4.1 of [Grau et al., 2011] below, which are essential in our analysis on the number of isolated zeros of $I(h)$.

Lemma 4.1. *Consider the Abelian integrals*

$$I_i(h) = \oint_{L_h} f_i(x)\, y^{2s-1} dx, \quad i = 0, 1, ..., n-1,$$

where L_h for each $h \in (0, h_0)$, is the level curve $\{A(x) + B(x)y^{2m} = h\}$, and define

$$l_i(x) := \frac{f_i(x)}{A'(x)} - \frac{f_i(z(x))}{A'(z(x))}.$$

Now suppose the followig conditions, are satisfied:
(i) $W[l_0, ..., l_i]$ is non-vanishing on $(0, x_r)$ for $i = 0, 1, ..., n-2$.
(ii) $W[l_0, ..., l_{n-1}]$ has k zeros on $(0, x_r)$ counted with multiplicities, and
(iii) $s > m(n + k - 2)$.
Then any nontrivial linear combination of $I_0, I_1, ..., I_{n-1}$ has at most $n + k - 1$ zeros on $(0, h_0)$ counted with multiplicities. Here, the notation $W[f_0, \cdots, f_n]$ stands for the Wronskians of the smooth functions $(f_0, \cdots, f_n)$.

Usually condition (iii) in Lemma 4.1 does not hold, and hence we can not apply Lemma 4.1 directly. To overcome this problem, we can use the next result (see [Grau et al., 2011], Lamma 4.1) to increase the power of y in the differential 1-form associated to the integral $I_i(h)$ defined above.

Lemma 4.2. *Let L_h be an oval with the level curve $A(x) + B(x)y^2 = h$ and consider a function F such that $\frac{F}{A'}$ is analytic at $x = 0$. Than, for any $k \in \mathbb{N}$,*

$$\oint_{L_h} F(x)y^{k-2}dx \quad = \quad \oint_{L_h} G(x)y^k dx$$

where $G(x) = \frac{2}{k}(\frac{BF}{A'})'(x) - (\frac{B'F}{A'})(x)$.

But in our case, $m = 1$, $n = 4$ and $s = 1$, therefore the condition $s > m(n+k-2)$ is not fulfilled even for $k = 1$. To overcome this problem, we will use Lemma 4.2 and will obtain new Abelian integrals for which the corresponding s is large enough to verify the inequality. Here, the power s has to be promoted to three so that the condition $s > n + k - 2$ holds.
On the oval L_h the following expression holds

$$\begin{aligned} I_i(h) &= \frac{1}{h}\oint_{L_h}\left(A(x) + \frac{y^2}{2}\right)x^i y dx \\ &= \frac{1}{2h}\left(\oint_{L_h} 2x^i A(x) y dx + \oint_{L_h} x^i y^3 dx\right), \qquad i = 0, 1, 3, 5. \end{aligned} \tag{14}$$

Now we apply Lemma 4.2 with $k = 3$ and $F(x) = 2x^i A(x)$ to the first integral above to get

$$\oint_{L_h} 2x^i A(x) y dx = \oint_{L_h} G_i(x) y^3 dx,$$

where $G_i(x) = \frac{d}{3dx}(\frac{2x^i A(x)}{A'(x)}) = -\frac{g_i(x)}{210(x-1)^4}$, with

$$g_i(x) \quad = \quad -35\,x^i\,(i+1) + 7\,x^{i+1}\,(14 + 17\,i) - 14\,x^{i+2}\,(9 + 11\,i) + 10\,x^{i+3}\,(8 + 9\,i) - 20\,x^{i+4}\,(i+1)\,.$$

By (14) we obtain that

$$\begin{aligned} I_i(h) &= \frac{1}{2h}\oint_{L_h}\left(x^i + G_i(x)\right)y^3 dx = \frac{1}{4h^2}\oint_{L_h}(2A(x) + y^2)(x^i + G_i(x))y^3 dx \\ &= \frac{1}{4h^2}\left(\oint_{L_h} 2(x^i + G_i(x))A(x)y^3 dx + \oint_{L_h}(x^i + G_i(x))y^5 dx\right). \end{aligned} \tag{15}$$

Again we apply Lemma 4.2 with $k = 5$ and $F(x) = 2(x^i + G_i(x))A(x)$ to the first integral above to get

$$\oint_{L_h} 2(x^i + G_i(x))A(x)y^3dx = \oint_{L_h} H_i(x)y^5dx,$$

where $H_i(x) = \frac{d}{5dx}(\frac{2(x^i+G_i(x))A(x)}{A'(x)}) = -\frac{h_i(x)}{73500(x-1)^8}$, *with*

$$\begin{aligned} h_i(x) &= -1225\,x^i\,(i+7)\,(i+1) + 245\,x^{i+1}\left(226 + 281\,i + 34\,i^2\right) - 49\,x^{i+2}\left(3394 + 4405\,i + 509\,i^2\right) \\ &+ 28\,x^{i+3}\left(10676 + 14027\,i + 1534\,i^2\right) - 14\,x^{i+4}\left(24857 + 32289\,i + 3324\,i^2\right) \\ &+ 70\,x^{i+5}\left(3806 + 4801\,i + 464\,i^2\right) - 20\,x^{i+6}\left(6503 + 7864\,i + 713\,i^2\right) \\ &+ 100\,x^{i+7}\left(368 + 423\,i + 36\,i^2\right) - 200\,x^{i+8}\,(2\,i+23)\,(i+1)\,. \end{aligned}$$

Now we can write

$$\begin{aligned} I_i(h) &= \frac{1}{4h^2}\left(\oint_{L_h}(x^i + G_i(x) + H_i(x))y^5dx\right) = \frac{1}{8h^3}\left(\oint_{L_h}(2A(x)+y^2)(x^i + G_i(x) + H_i(x))y^5dx\right) \\ &= \frac{1}{8h^3}\left(\oint_{L_h}2(x^i + G_i(x) + H_i(x))A(x)y^5dx + \oint_{L_h}(x^i + G_i(x) + H_i(x))y^7dx\right). \end{aligned} \tag{16}$$

Again we will apply Lemma 4.2 with $k = 7$ and $F(x) = 2(x^i+G_i(x)+H_i(x))A(x)$ to the first integral above to get

$$\oint_{L_h} 2(x^i + G_i(x) + H_i(x))A(x)y^5dx = \oint_{L_h} K_i(x)y^7dx,$$

where $K_i(x) = \frac{d}{7dx}(\frac{2(x^i+G_i(x)+H_i(x))A(x)}{A'(x)}) = \frac{k_i(x)}{36015000(x-1)^12}$, *with*

$$\begin{aligned} k_i(x) &= 42875\,x^i\,(i+11)\,(i+7)\,(i+1) - 25725\,x^{i+1}\left(1310 + 1703\,i + 333\,i^2 + 17\,i^3\right) \\ &+ 1715\,x^{i+2}\left(95058 + 127822\,i + 24283\,i^2 + 1197\,i^3\right) \\ &- 49\,x^{i+3}\left(10021984 + 13732156\,i + 2523089\,i^2 + 119681\,i^3\right) \\ &+ 98\,x^{i+4}\left(10417314 + 14366612\,i + 2544727\,i^2 + 115869\,i^3\right) \\ &- 294\,x^{i+5}\left(5227940 + 7183236\,i + 1223831\,i^2 + 53403\,i^3\right) \\ &+ 56\,x^{i+6}\left(30586815 + 41522047\,i + 6794274\,i^2 + 283844\,i^3\right) \\ &- 140\,x^{i+7}\left(10141200 + 13509256\,i + 2121113\,i^2 + 84801\,i^3\right) \\ &+ 420\,x^{i+8}\left(2063665 + 2682949\,i + 404038\,i^2 + 15458\,i^3\right) \\ &- 100\,x^{i+9}\left(3793174 + 4792425\,i + 692174\,i^2 + 25350\,i^3\right) \\ &+ 200\,x^{i+10}\left(563877 + 690095\,i + 95614\,i^2 + 3354\,i^3\right) \\ &- 1000\,x^{i+11}\left(20424 + 24155\,i + 3212\,i^2 + 108\,i^3\right) \\ &+ 2000\,x^{i+12}\,(2\,i+23)\,(2\,i+37)\,(i+1)\,. \end{aligned}$$

Finally we conclude that

$$8h^3I_i(h) = \oint_{L_h} f_i(x)y^7dx \equiv \tilde{I}_i(h), \tag{17}$$

where $f_i(x) = x^i + G_i(x) + H_i(x) + K_i(x)$. It is clear that $\{I_0, I_1, I_3, I_5\}$ is an extended complete Chebyshev system with accuracy 1 on $(0, \frac{1}{140})$ if and only if $\{\tilde{I}_0, \tilde{I}_1, \tilde{I}_3, \tilde{I}_5\}$ is as well. Now we can *apply Lemma 4.1, since $s = 4$ and hence the condition $s > m(n + k - 2)$ holds when $k = 1$. Thus, by setting*

$$\ell_i(x) := \left(\frac{f_i}{A'}\right)(x) - \left(\frac{f_i}{A'}\right)(z(x)),$$

it is needed to check that $\{\ell_0, \ell_1, \ell_3, \ell_5\}$ is a complete Chebyshev system of accuracy 1 on $x \in (0, 1)$. As a matter of fact, by proving the next lemma, it will be shown that $\{\ell_0, \ell_1, \ell_3, \ell_5\}$ is an extended complete Chebyshev system with accuracy 1.

Lemma 4.3. *It holds that*

(*i*) $W[\ell_0](x) \neq 0$ *for all* $x \in (0, 1)$;
(*ii*) $W[\ell_0, \ell_1](x) \neq 0$ *for all* $x \in (0, 1)$;
(*iii*) $W[\ell_0, \ell_1, \ell_3](x) \neq 0$ *for all* $x \in (0, 1)$;
(*iv*) $W[\ell_0, \ell_1, \ell_3, \ell_5](x) \neq 0$ *for all* $x \in (0, 1)\backslash\{x^*\}$,

where $x^* \in (0, 1)$ will be introduced in the proof.

Proof. By an easy computation, it is easily seen that the involution $z = z(x)$ for $x \in (0, 1)$ satisfies

$$A(x) - A(z) \quad = \quad -\frac{1}{140}(x - z)q(x, z) = 0,$$

where

$$\begin{aligned} q(x, z) \quad = \quad & 20\,x^6 - 70\,x^5 + 20\,zx^5 + 84\,x^4 - 70\,zx^4 + 20\,z^2x^4 - 35\,x^3 + 84\,zx^3 - 70\,z^2x^3 \\ + \quad & 20\,x^3z^3 - 35\,zx^2 + 84\,z^2x^2 - 70\,x^2z^3 + 20\,x^2z^4 - 35\,z^2x + 84\,xz^3 - 70\,xz^4 \\ + \quad & 20\,xz^5 - 35\,z^3 + 84\,z^4 - 70\,z^5 + 20\,z^6. \end{aligned}$$

By using Maple 18, it is founded that

$$\begin{aligned} W[\ell_0](x) \quad &= \quad \frac{(x - z)W_0(x, z)}{12005000\,(x - 1)^{15}\,x^3\,(z - 1)^{15}\,z^3}, \\ W[\ell_0, \ell_1](x) \quad &= \quad -\frac{(x - z)^3W_1(x, z)}{144120025000000z^6\,(z - 1)^{30}\,x^6\,(x - 1)^{30}\,W_{01}(x, z)}, \\ W[\ell_0, \ell_1, \ell_3](x) \quad &= \quad -\frac{3(x - z)^6W_2(x, z)}{86508045006250000000z^8\,(z - 1)^{45}\,x^8\,(x - 1)^{45}\,W_{01}^3(x, z)}, \\ W[\ell_0, \ell_1, \ell_3, \ell_5](x) \quad &= \quad \frac{3(x - z)^{10}W_3(x, z)}{519264540150015625000000000\,(z - 1)^{59}\,z^{10}\,(x - 1)^{59}\,x^{10}W_{01}^6(x, z)}, \end{aligned}$$

where $W_i(x, z),\ i = 0, 1, 2$ are polynomials with long expressions in (x, z) and

$$\begin{aligned} W_{01}(x, z) \quad = \quad & 20\,x^5 - 70\,x^4 + 84\,x^3 - 35\,x^2 - 105\,z^2 + 336\,z^3 - 350\,z^4 + 120\,z^5 + 40\,zx^4 - 140\,zx^3 \\ + \quad & 60\,z^2x^3 + 168\,zx^2 - 210\,z^2x^2 + 80\,x^2z^3 - 70\,zx + 252\,z^2x - 280\,xz^3 + 100\,xz^4. \end{aligned}$$

The resultant with respect to z between $W_{01}(x, z)$ and $q(x, z)$ is

$$p_{01}(x) = 53782400000\,x^6\left(20\,x^3 - 70\,x^2 + 84\,x - 35\right)^3\left(20\,x^3 + 10\,x^2 + 4\,x + 1\right)^3(x - 1)^6\,.$$

By using Maple 18, it is easy to see that $p_{01}(x)$ does not have any zeros in $(0, 1)$. This implies that $W[\ell_0, \ell_1]$, $W[\ell_0, \ell_1, \ell_3]$ and $W[\ell_0, \ell_1, \ell_3, \ell_5]$ are well defined in the domain $x_l < z < 0 < x < 1$.

In order to determine if these four Wronskians have zeros on $(0,1)$, we shall rely on the symbolic computations by Maple 18 under Linux to compute the resultant between $W_i(x,z),\ i=0,1,2,3$ and $q(x,z)$ with respect to z, and then we will apply Sturm's Theorem to assert nonexistence of zeros of $p_i(x),\ i=0,1,2,3$, in $(0,1)$ where $p_i(x)$ are polynomials of high degree in x that will be described below.

Case (i). The resultant with respect to z between $q(x,z)$ and $W_0(x,z)$ is $R_0(x)=x^6\,(x-1)^{42}\,p_0(x)$, where $p_0(x)$ is a polynomial of degree 126 in x. By applying Sturm's Theorem we get that $p_0(x)\neq 0$ for all $x\in(0,1)$. Thus, $W_0(x,z)=0$ and $q(x,z)=0$ have no common roots. This fact implies that $W[\ell_0](x)\neq 0$ for all $x\in(0,1)$.

Case (ii). The resultant with respect to z between $q(x,z)$ and $W_1(x,z)$ is $R_1(x)=x^{16}\,(x-1)^{88}\,p_1(x)$, where $p_1(x)$ is a polynomial of degree 268 in x. By applying Sturm's Theorem we get that $p_1(x)\neq 0$ for all $x\in(0,1)$. Thus, $W_1(x,z)=0$ and $q(x,z)=0$ have no common roots. This fact implies that $W[\ell_0,\ell_1](x)\neq 0$ for all $x\in(0,1)$.

Case (iii). The resultant with respect to z between $q(x,z)$ and $W_2(x,z)$ is $R_2(x)=x^{28}\,(x-1)^{138}\,p_2(x)$, where $p_2(x)$ is a polynomial of degree 422 in x. By applying Sturm's Theorem we see that $p_2(x)$ has a unique zero in the open interval $(0,1)$. Therefore the method used in cases (i) and (ii) fails to work in this case. In order to make sure if $W_2(x,z)$ and $q(x,z)$ have common roots, we use the direct program with Maple 18 (under Linux) to find all the intervals in which all the common roots of $W_2(x,z)$ and $q(x,z)$ on the whole plane may occur.

```
> with(RegularChains):
> with(ChainTools):
> with(SemiAlgebraicSetTools):
> R := PolynomialRing([x,z]):
> sys := [w_2(x, z), q(x, z)]:
> dec := Triangularize(sys, R);
          [regular_chain, regular_chain, regular_chain]
> L := map(Equations, dec, R);
```

$$[[k_1(x,z),k_2(z)],[x-1,z-1],[x,z]]$$

where $k_1(x,z)=k_{11}(z)x+k_{12}(z)$, k_{11} is a polynomial in z of degree 421, k_{12} is a polynomial in z of degree 421 and k_2 is a polynomial in z of degree 422. It is obvious that the second regular chain and the third one do not have roots satisfying $x_l<z<0<x<1$ where $x_l\approx -0.3423840949$, and the first regular chain $[k_1(x,z),k_2(z)]$ is square-free and zero-dimensional (because the number of variables equals the number of polynomials). $L[1][1]$ and $L[1][2]$ represent k_1 and k_2 in Maple and they have two and one variables, respectively. Therefore we need to change their order in the first regular chain.

```
> C := Chain([L[1][2], L[1][1]], Empty(R), R);
                             regular_chain
> RL := RealRootIsolate(C, R, 'abserr' = 1/10^5);
                           [box, box, box, box, box, box]
>evalf(map(BoxValues, RL, R));
[[x = [1.332499319, 1.332499319], z = [-.2175672857, -.2175672857]],
 [x = [1.188453671, 1.188453671], z = [-.3372832568, -.3372832568]],
 [x = [-.2175672857, -.2175672856], z = [1.332499319, 1.332499319]],
 [x = [.5846840049, .5846840050], z = [1.267152074, 1.267152074]],
 [x = [-.3372832568, -.3372832568], z = [1.188453671, 1.188453671]],
 [x = [1.267152074, 1.267152074], z = [.5846840050, .5846840050]]].
```

From the result of the program, we see that there are 6 pairs of common roots of $W_2(x,z)$ and $q(x,z)$ in the above mentioned intervals, but no pairs satisfy $x_l<z<0<x<1$. This fact implies that

$W[\ell_0, \ell_1, \ell_3](x) \neq 0$ for all $x \in (0,1)$.

Case (iv). *The resultant with respect to z between $q(x,z)$ and $W_3(x,z)$ is $R_3(x) = x^{44}\,(x-1)^{188}\,p_3(x)$, where $p_3(x)$ is a polynomial of degree 590 in x. By applying Sturm's Theorem we find that $p_3(x)$ has a unique zero in the interval $(0,1)$. In order to make sure if $W_3(x,z)$ and $q(x,z)$ have common roots, we use the direct program with Maple 18 as before to discover all the intervals in which all the common roots of $W_3(x,z)$ and $q(x,z)$ on the whole plane exist.*

```
> with(RegularChains):
> with(ChainTools):
> with(SemiAlgebraicSetTools):
> R := PolynomialRing([x,z]):
> sys := [w_3(x, z), q(x, z)]:
> dec := Triangularize(sys, R);
          [regular_chain, regular_chain, regular_chain]
> L := map(Equations, dec, R);
```

$$[[r_1(x,z), r_2(z)], [x-1, z-1], [x, z]]$$

where $r_1(x,z) = r_{11}(z)x + r_{12}(z)$, r_{11} is a polynomial in z of degree 589, r_{12} is a polynomial in z of degree 589 and r_2 is a polynomial in z of degree 590. It is obvious that the second regular chain and the third one do not have roots satisfying $x_l < z < 0 < x < 1$ where $x_l \approx -.3423840949$, and the first regular chain $[r_1(x,z), r_2(z)]$ is square-free and zero-dimensional (because the number of variables equals the number of polynomials). $L[1][1]$ and $L[1][2]$ represent r_1 and r_2 in Maple and they have two and one variables, respectively. Therefore we need to change their order in the first regular chain.

```
> C := Chain([L[1][2], L[1][1]], Empty(R), R);
                          regular_chain
> RL := RealRootIsolate(C, R, 'abserr' = 1/10^5);
                       [box, box, box, box, box, box]
>evalf(map(BoxValues, RL, R));
[[x = [1.342226771, 1.342226771], z = [-0.8416542251, -0.8416542251]],
[x = [.7968688606, .7968688607], z = [-.3397589321, -.3397589321]],
[x = [-0.8416542254, -0.8416542248], z = [1.342226771, 1.342226771]],
[x = [-.3397589321, -.3397589321], z = [.7968688606, .7968688606]]].
```

As a result, we see that there are 4 pairs of common roots of $W_3(x,z)$ and $q(x,z)$ in the above mentioned intervals, but exactly one pair satisfies $x_l < z < 0 < x < 1$. Therefore, there is a unique $x^* = 0.796868861 \in (0,1)$ such that $W[\ell_0, \ell_1, \ell_3, \ell_5](x^*) = 0$. Hence the proof is complete. □

Corollary 4.4. *The exact upper bound for the maximal number of isolated zeros of $I(h,\delta)$, defined in* (4), *is 4 on the open interval $(0, \frac{1}{140})$. Hence, the Liénard system (H_ε) can have at most 4 limit cycles bifurcating from the corresponding period annulus.*

Proof. It follows from Lemma 4.1 and Lemma 4.3 that $\{I_0(h), I_1(h), I_3(h), I_5(h)\}$ is a Chebyshev system with accuracy 1 on $(0, \frac{1}{140})$. In consequence, there are at most 4 zeros of $I(h)$ on $(0, \frac{1}{140})$. Therefore, there are at most 4 limit cycles of system (H_ε) bifurcated from the period annulus $\{L_h\}_h$ of system (H_0). □

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